Blocking Dominating Sets for H-Free Graphs via Edge Contractions

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Abstract

In this paper, we consider the following problem: given a connected graph G, can we reduce the domination number of G by one by using only one edge contraction? We show that the problem is NP-hard when restricted to $\{P_6, P_4 + P_2\}$-free graphs and that it is coNP-hard when restricted to subcubic claw-free graphs and $2P_3$-free graphs. As a consequence, we are able to establish a complexity dichotomy for the problem on H-free graphs when H is connected.

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1 Introduction

A *blocker problem* asks whether given a graph G, a graph parameter π, a set $\mathcal{O}$ of one or more graph operations and an integer $k \geq 1$, G can be transformed into a graph G' by using at most k operations from $\mathcal{O}$ such that $\pi(G') \leq \pi(G) - d$ for some *threshold* $d \geq 0$. Such a designation follows from the fact that the set of vertices or edges involved can be viewed as "blocking" the parameter π. Identifying such sets may provide information on the structure of the input graph; for instance, if $\pi = \alpha$, $k = d = 1$ and $\mathcal{O} = \{\text{vertex deletion}\}$, the problem is equivalent to testing whether the input graph contains a vertex that is in every maximum independent set (see [18]). Blocker problems have received much attention in the recent literature (see for instance [1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19]) and have been related to other well-known graph problems such as Hadwiger Number, Club Contraction and several graph transversal problems (see for instance [7, 17]). The graph parameters mainly considered in the literature so far include the chromatic number, the independence number, the clique number, the matching number and the vertex cover number while the set $\mathcal{O}$ is always a singleton consisting of a vertex deletion, edge contraction, edge deletion or edge addition. In this paper, we focus on the domination number γ, let $\mathcal{O}$ consist of an edge contraction and set the threshold d to one.

Formally, let $G = (V, E)$ be a graph. The *contraction* of an edge $uv \in E$ removes vertices u and v from G and replaces them by a new vertex that is made adjacent to precisely those vertices which were adjacent to u or v in G (without introducing self-loops nor multiple edges). We say that a graph G can be k-*contracted* into a graph G', if G can be transformed into G' by a sequence of at most k edge contractions, for an integer $k \geq 1$. The problem we consider is then the following (note that contracting an edge cannot increase the domination number).

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k-Edge Contraction(γ)	
Instance:	A connected graph $G = (V, E)$.
Question:	Can G be k-contracted into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$?

Reducing the domination number using edge contractions was first considered in [10]. The authors proved that for a connected graph G such that $\gamma(G) \geq 2$, we have $ct_\gamma(G) \leq 3$, where $ct_\gamma(G)$ denotes the minimum number of edge contractions required to transform G into a graph G' such that $\gamma(G') \leq \gamma(G) - 1$ (note that if $\gamma(G) = 1$ then G is a No-instance for k-Edge Contraction(γ) independently of the value of k). Thus, if G is a connected graph with $\gamma(G) \geq 2$, then G is always a Yes-instance for k-Edge Contraction(γ) when $k \geq 3$. It was later shown in [9] that k-Edge Contraction(γ) is coNP-hard for $k \leq 2$ and so, restrictions on the input graph to some special graph classes were considered. In particular, the authors in [9] proved that for $k = 1, 2$, the problem is polynomial-time solvable for P_5-free graphs while for $k = 1$, it remains NP-hard when restricted to P_9-free graphs and $\{C_3, \ldots, C_\ell\}$-free graphs, for any $\ell \geq 3$.

In this paper, we continue the systematic study of the computational complexity of 1-Edge Contraction(γ) initiated in [9]. Ultimately, the aim is to obtain a complete classification for 1-Edge Contraction(γ) restricted to H-free graphs, for any (not necessarily connected) graph H, as it has been done for other blocker problems (see for instance [8, 18, 19]). As a step towards this end, we prove the following three theorems.

▶ **Theorem 1.** 1-Edge Contraction(γ) *is* NP*-hard when restricted to* $\{P_6, P_4 + P_2\}$*-free graphs.*

▶ **Theorem 2.** 1-Edge Contraction(γ) *is* coNP*-hard when restricted to subcubic claw-free graphs.*

▶ **Theorem 3.** 1-Edge Contraction(γ) *is* coNP*-hard when restricted to* $2P_3$*-free graphs.*

Note that Theorems 1 and 2 lead to a complexity dichotomy for H-free graphs when H is connected. Indeed, since 1-Edge Contraction(γ) is NP-hard when restricted to $\{C_3, \ldots, C_\ell\}$-free graphs, for any $\ell \geq 3$, it follows that 1-Edge Contraction(γ) is NP-hard for H-free graphs when H contains a cycle. If H is a tree with a vertex of degree at least three, we conclude by Theorem 2 that 1-Edge Contraction(γ) is coNP-hard for H-free graphs; and Theorem 1 shows that if H is a path of length at least 6, then 1-Edge Contraction(γ) is NP-hard for H-free graphs. Finally, since in [9] 1-Edge Contraction(γ) is shown to be polynomial-time solvable on $\{P_5 + pK_1\}$-free graphs for any $p \geq 0$, it follows that 1-Edge Contraction(γ) is polynomial-time solvable on H-free graphs if $H \subseteq_i P_5$. We therefore obtain the following result.

▶ **Corollary 4.** *Let H be a connected graph. If $H \subseteq_i P_5$ then* 1-Edge Contraction(γ) *is polynomial-time solvable on H-free graphs, otherwise it is* NP*-hard or* coNP*-hard.*

If the graph H is not required to be connected, we know the following. As previously mentioned, 1-Edge Contraction(γ) is NP-hard (resp. coNP-hard) on H-free graphs when H contains a cycle (resp. an induced claw). Thus, there remains to consider the case where H is a linear forest, that is, a disjoint union of paths. Theorems 1 and 3 show that if H contains either a P_6, a $P_4 + P_2$ or a $2P_3$ as an induced subgraph, then 1-Edge Contraction(γ) is NP-hard or coNP-hard on H-free graphs. Since it is known that 1-Edge Contraction(γ) is polynomial-time solvable on H-free graphs if $H \subseteq_i P_5 + pK_1$, there remains to determine the complexity status of the problem restricted to H-free graphs when $H = P_3 + qP_2 + pK_1$, for $q \geq 1$ and $p \geq 0$.

2 Preliminaries

Throughout the paper, we only consider finite, undirected and connected graphs that have no self-loops or multiple edges. We refer the reader to [6] for any terminology and notation not defined here.

For $n \geq 1$, the path and cycle on n vertices are denoted by P_n and C_n respectively. The *claw* is the complete bipartite graph with one partition of size one and the other of size three.

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$, and let $u \in V(G)$. We denote by $N_G(u)$, or simply $N(u)$ if it is clear from the context, the set of vertices that are adjacent to u i.e., the *neighbors* of u, and let $N[u] = N(u) \cup \{u\}$. The *degree* of a vertex u, denoted by $d_G(u)$ or simply $d(u)$ if it is clear from the context, is the size of its neighborhood i.e., $d(u) = |N(u)|$. The maximum degree in G is denoted by $\Delta(G)$ and G is *subcubic* if $\Delta(G) \leq 3$.

For a family $\{H_1, \ldots, H_p\}$ of graphs, G is said to be $\{H_1, \ldots, H_p\}$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of $\{H_1\}$-free. For a subset $V' \subseteq V(G)$, we let $G[V']$ denote the subgraph of G *induced* by V', which has vertex set V' and edge set $\{uv \in E(G) \mid u, v \in V'\}$.

A subset $S \subseteq V(G)$ is called an *independent set* or is said to be *independent*, if no two vertices in S are adjacent. A subset $D \subseteq V(G)$ is called a *dominating set*, if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D; the *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set. For any $v \in D$ and $u \in N[v]$, v is said to *dominate* u (in particular, v dominates itself). We say that D *contains an edge* (or more) if the graph $G[D]$ contains an edge (or more). A dominating set D of G is *efficient* if for every vertex $v \in V$, $|N[v] \cap D| = 1$ that is, v is dominated by exactly one vertex.

In the following, we consider those graphs for which one edge contraction suffices to decrease their domination number by one. A characterization of this class is given in [10].

▶ **Theorem 5** ([10]). *For a connected graph G, $ct_\gamma(G) = 1$ if and only if there exists a minimum dominating set in G that is not independent.*

In order to prove Theorems 2 and 3, we introduce the following two problems.

ALL EFFICIENT MD	
Instance:	A connected graph $G = (V, E)$.
Question:	Is every minimum dominating set of G efficient?

ALL INDEPENDENT MD	
Instance:	A connected graph $G = (V, E)$.
Question:	Is every minimum dominating set of G independent?

The following is then a straightforward consequence of Theorem 5.

▷ Fact 1. Given a graph G, G is a YES-instance for 1-EDGE CONTRACTION(γ) if and only if G is a NO-instance for ALL INDEPENDENT MD.

3 The proof of Theorem 1

In this section, we show that 1-EDGE CONTRACTION(γ) is NP-hard when restricted to $\{P_6, P_4 + P_2\}$-free graphs.

To this end, we give a reduction from Dominating Set. Given an instance (G, ℓ) for Dominating Set, we construct an instance G' for 1-Edge Contraction(γ) as follows. We denote by $\{v_1, \ldots, v_n\}$ the vertex set of G. The vertex set of the graph G' is given by $V(G') = V_0 \cup \ldots \cup V_\ell \cup \{x_0, \ldots, x_\ell, y\}$, where each V_i is a copy of the vertex set of G. We denote the vertices of V_i by $v^i_1, v^i_2, \ldots, v^i_n$. The adjacencies in G' are then defined as follows:

- $V_0 \cup \{x_0\}$ is a clique;
- $yx_0 \in E(G')$;

and for any $1 \leq i \leq \ell$,

- V_i is an independent set;
- x_i is adjacent to all the vertices in $V_0 \cup V_i$;
- v^i_j is adjacent to $\{v^0_a \mid v_a \in N_G[v_j]\}$ for any $1 \leq j \leq n$.

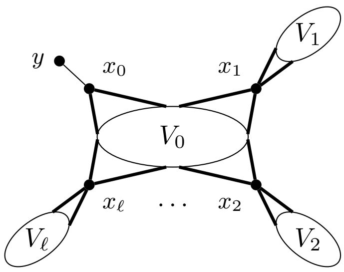


Figure 1 The graph G' (thick lines indicate that the vertex x_i is adjacent to every vertex in V_0 and V_i, for $i = 0, \ldots, \ell$).

▷ Claim 1. $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$.

Proof. It is clear that $\{x_0, x_1, \ldots, x_\ell\}$ is a dominating set of G'; thus, $\gamma(G') \leq \ell+1$. If $\gamma(G) \leq \ell$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, it is easily seen that $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_0\}$ is a dominating set of G'. Thus, $\gamma(G') \leq \gamma(G) + 1$ and so, $\gamma(G') \leq \min\{\gamma(G) + 1, \ell + 1\}$. Now, suppose to the contrary that $\gamma(G') < \min\{\gamma(G) + 1, \ell + 1\}$ and consider a minimum dominating set D' of G'. We first make the following simple observation.

▷ Observation 1. For any dominating set D of G', $D \cap \{y, x_0\} \neq \emptyset$.

Now, since $\gamma(G') < \ell + 1$, there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$ (otherwise, $\{x_1, \ldots, x_\ell\} \subset D'$ and combined with Observation 1, D' would be of size at least $\ell + 1$). But then, $D'' = D' \cap V_0$ must dominate every vertex in V_i, and so $|D''| \geq \gamma(G)$. Since $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$), we then have $\gamma(G) \leq |D'| - 1$, a contradiction. Thus, $\gamma(G') = \min\{\gamma(G) + 1, \ell + 1\}$. ◁

We now show that (G, ℓ) is a Yes-instance for Dominating Set if and only if G' is a Yes-instance for 1-Edge Contraction(γ).

Assume first that $\gamma(G) \leq \ell$. Then $\gamma(G') = \gamma(G) + 1$ by the previous claim, and if $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimum dominating set of G, then $\{v^0_{i_1}, \ldots, v^0_{i_k}, x_0\}$ is a minimum dominating set of G' which is not independent. Hence, by Theorem 5, G' is a Yes-instance for 1-Edge Contraction(γ).

Conversely, assume that G' is a Yes-instance for 1-Edge Contraction(γ) i.e., there exists a minimum dominating set D' of G' which is not independent (see Theorem 5). Then, Observation 1 implies that there exists $1 \leq i \leq \ell$ such that $x_i \notin D'$; indeed, if it weren't the case, by Claim 1 we would then have $\gamma(G') = \ell + 1$ and thus, D' would consist of $x_1, \ldots, x_\ell$

and either y or x_0. In both cases, D' would be independent, a contradiction. It follows that $D'' = D' \cap V_0$ must dominate every vertex in V_i and thus, $|D''| \geq \gamma(G)$. But $|D''| \leq |D'| - 1$ (recall that $D' \cap \{y, x_0\} \neq \emptyset$) and so by Claim 1, $\gamma(G) \leq |D'| - 1 \leq (\ell + 1) - 1$ that is, (G, ℓ) is a Yes-instance for Dominating Set.

We next show that G' is a P_6-free graph. Let P be an induced path of G'. First observe that since V_0 is a clique, $|V(P) \cap V_0| \leq 2$. If $|V(P) \cap V_0| = 0$, since each V_i is independent and the same holds for $\{x_0, \ldots, x_\ell\}$, we have that $|V(P)| \leq 3$. We now consider the following two cases.

Case 1. $|V(P) \cap V_0| = 2$. Let $u, v \in V_0$ be the vertices of $V(P) \cap V_0$. Since P is an induced path, u and v appear consecutively in P, that is, $uv \in E(P)$. Furthermore, $V(P) \cap \{x_0, \ldots, x_\ell\} = \emptyset$ since u and v are adjacent to all the vertices of $\{x_0, \ldots, x_\ell\}$. If u has another neighbor $w \in V_i$ in P, for some $i > 0$, then since $N(w) \subset V_0 \cup \{x_i\}$, w can have no neighbor in P other than u, that is, w is an endpoint of the path. Symmetrically, the same holds for a neighbor of v in P different from u. Hence, we conclude that $|V(P)| \leq 4$.

Case 2. $|V(P) \cap V_0| = 1$. Let $u \in V_0$ be the vertex of $V(P) \cap V_0$. If $V(P) \cap \{x_0, \ldots, x_\ell\} = \emptyset$, then it is easy to see that $|V(P)| \leq 3$, since any neighbor of u in the path must belong to $\cup_{1 \leq i \leq \ell} V_i$ and, by the same argument as in Case 1, such a neighbor would have to be an endpoint of the path. If $V(P) \cap \{x_0, \ldots, x_\ell\} \neq \emptyset$, let x_i be a vertex that is in P. Since $ux_i \in E(G')$, we necessarily have that $ux_i \in E(P)$. Suppose that x_i has another neighbor w in P. Then $w \in V_i$ since $N(x_i) = V_0 \cup V_i$. By the argument used above, w must then be an endpoint of the path; and since u can have at most two neighbors in $\{x_0, \ldots, x_\ell\}$, we conclude that $|V(P)| \leq 5$.

Finally, to see that G' is also a $\{P_4 + P_2\}$-free graph, it suffices to note that any induced P_4 of G' contains at least one vertex of the clique V_0. This concludes the proof of Theorem 1.

4 The proof of Theorem 2

In this section, we show that 1-Edge Contraction(γ) is coNP-hard when restricted to subcubic claw-free graphs. To this end, we first prove the following.

▶ **Lemma 6.** All Efficient MD *is* NP*-hard when restricted to subcubic graphs.*

Proof. We reduce from Positive Exactly 3-Bounded 1-In-3 3-Sat, where each variable appears in exactly three clauses and only positively, each clause contains three positive literals, and we want a truth assignment such that each clause contains exactly one true literal. This problem is shown to be NP-complete in [14]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Efficient MD as follows. For any variable $x \in X$, we introduce a copy of C_9, which we denote by G_x, with three distinguished *true vertices* T^1_x, T^2_x and T^3_x, and three distinguished *false vertices* F^1_x, F^2_x and F^3_x (see Fig. 2a). For any clause $c \in C$ containing variables x_1, x_2 and x_3, we introduce the gadget G_c depicted in Fig. 2b which has one distinguished *clause vertex* c and three distinguished *variable vertices* x_1, x_2 and x_3 (note that G_c is not connected). For every $j \in \{1, 2, 3\}$, we then add an edge between x_j and $F^i_{x_j}$ and between c and $T^i_{x_j}$ for some $i \in \{1, 2, 3\}$ so that $F^i_{x_j}$ (resp. $T^i_{x_j}$) is adjacent to exactly one variable vertex (resp. clause vertex). We denote by G_Φ the resulting graph. Note that $\Delta(G_\Phi) = 3$.

▷ Observation 1. For any dominating set D of G_Φ, $|D \cap V(G_x)| \geq 3$ for any $x \in X$ and $|D \cap V(G_c)| \geq 1$ for any $c \in C$. In particular, $\gamma(G_\Phi) \geq 3|X| + |C|$.

(a) The variable gadget G_x. **(b)** The clause gadget G_c.

Figure 2 Construction of the graph G_Φ (the rectangle indicates that the corresponding set of vertices induces a clique).

Indeed, for any $x \in X$, since u_x^1, u_x^2 and u_x^3 must be dominated and their neighborhoods are pairwise disjoint and contained in G_x, it follows that $|D \cap V(G_x)| \geq 3$. For any $c \in C$, since the vertices of K_c must be dominated and their neighborhoods are contained in G_c, $|D \cap V(G_c)| \geq 1$. ⌟

▷ Observation 2. For any $x \in X$, if D is a minimum dominating set of G_x then either $D = \{u_x^1, u_x^2, u_x^3\}$, $D = \{T_x^1, T_x^2, T_x^3\}$ or $D = \{F_x^1, F_x^2, F_x^3\}$.

▷ Claim 1. Φ is satisfiable if and only if $\gamma(G_\Phi) = 3|X| + |C|$.

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. For any variable $x \in X$, if x is true, add T_x^1, T_x^2 and T_x^3 to D; otherwise, add F_x^1, F_x^2 and F_x^3 to D. For any clause $c \in C$ containing variables x_1, x_2 and x_3, exactly one variable is true, say x_1 without loss of generality; we then add $l_{\{x_1\}}$ to D. Clearly, D is dominating and we conclude by Observation 1 that $\gamma(G_\Phi) = 3|X| + |C|$.

Conversely, assume that $\gamma(G_\Phi) = 3|X| + |C|$ and consider a minimum dominating set D of G_Φ. Then by Observation 1, $|D \cap V(G_x)| = 3$ for any $x \in X$ and $|D \cap V(G_c)| = 1$ for any $c \in C$. Now, for a clause $c \in C$ containing variables x_1, x_2 and x_3, if $D \cap \{c, x_1, x_2, x_3\} \neq \emptyset$ then $D \cap V(K_c) = \emptyset$ and so, at least two vertices from K_c are not dominated; thus, $D \cap \{c, x_1, x_2, x_3\} = \emptyset$. It follows that for any $x \in X$, $D \cap V(G_x)$ is a minimum dominating set of G_x which by Observation 2 implies either $\{T_x^1, T_x^2, T_x^3\} \subset D$ or $D \cap \{T_x^1, T_x^2, T_x^3\} = \emptyset$; and we conclude similarly that either $\{F_x^1, F_x^2, F_x^3\} \subset D$ or $D \cap \{F_x^1, F_x^2, F_x^3\} = \emptyset$. Now given a clause $c \in C$ containing variables x_1, x_2 and x_3, since $D \cap \{c, x_1, x_2, x_3\} = \emptyset$, at least one true vertex adjacent to the clause vertex c must belong to D, say $T_{x_1}^i$ for some $i \in \{1, 2, 3\}$ without loss of generality. It then follows that $\{T_{x_1}^1, T_{x_1}^2, T_{x_1}^3\} \subset D$ and $D \cap \{F_{x_1}^1, F_{x_1}^2, F_{x_1}^3\} = \emptyset$ which implies that $l_{\{x_1\}} \in D$ (either x_1 or a vertex from K_c would otherwise not be dominated). But then, since x_j for $j \neq 1$, must be dominated, it follows that $\{F_{x_j}^1, F_{x_j}^2, F_{x_j}^3\} \subset D$. We thus construct a truth assignment satisfying Φ as follows: for any variable $x \in X$, if $\{T_x^1, T_x^2, T_x^3\} \subset D$, set x to true, otherwise set x to false. ◁

▷ Claim 2. $\gamma(G_\Phi) = 3|X| + |C|$ if and only if every minimum dominating set of G_Φ is efficient.

Proof. Assume that $\gamma(G_\Phi) = 3|X| + |C|$ and consider a minimum dominating set D of G_Φ. Then by Observation 1, $|D \cap V(G_x)| = 3$ for any $x \in X$ and $|D \cap V(G_c)| = 1$ for any $c \in C$. As shown previously, it follows that for any clause $c \in C$ containing variables x_1, x_2 and x_3, $D \cap \{c, x_1, x_2, x_3\} = \emptyset$; and for any $x \in X$, either $\{T_x^1, T_x^2, T_x^3\} \subset D$ or $D \cap \{T_x^1, T_x^2, T_x^3\} = \emptyset$ (we conclude similarly with $\{F_x^1, F_x^2, F_x^3\}$ and $\{u_x^1, u_x^2, u_x^3\}$). Thus, for any $x \in X$, every vertex in G_x is dominated by exactly one vertex. Now given a clause $c \in C$ containing variables x_1, x_2 and x_3, since the clause vertex c does not belong to D, there exists at least one true vertex adjacent to c which belongs to D. Suppose to the contrary that c has strictly more than one neighbor in D, say $T^i_{x_1}$ and $T^j_{x_2}$ without loss of generality. Then, $\{T^1_{x_k}, T^2_{x_k}, T^3_{x_k}\} \subset D$ for $k = 1, 2$ which implies that $D \cap \{F^1_{x_1}, F^2_{x_1}, F^3_{x_1}, F^1_{x_2}, F^2_{x_2}, F^3_{x_2}\} = \emptyset$ as $|D \cap V(G_{x_k})| = 3$ for $k = 1, 2$. It follows that the variable vertices x_1 and x_2 must be dominated by some vertices in G_c; but $|D \cap V(G_c)| = 1$ and $N[x_1] \cap N[x_2] = \emptyset$ and so, either x_1 or x_2 is not dominated. Thus, c has exactly one neighbor in D, say $T^i_{x_1}$ without loss of generality. Then, necessarily $D \cap V(G_c) = \{l_{\{x_1\}}\}$ for otherwise either x_1 or some vertex in K_c would not be dominated. But then, it is clear that every vertex in G_c is dominated by exactly one vertex; thus, D is efficient.

Conversely, assume that every minimum dominating set of G_Φ is efficient and consider a minimum dominating set D of G_Φ. If for some $x \in X$, $|D \cap V(G_x)| \geq 4$, then clearly at least one vertex in G_x is dominated by two vertices in $D \cap V(G_x)$. Thus, $|D \cap V(G_x)| \leq 3$ for any $x \in X$ and we conclude by Observation 1 that in fact, equality holds. The next observation immediately follows from the fact that D is efficient.

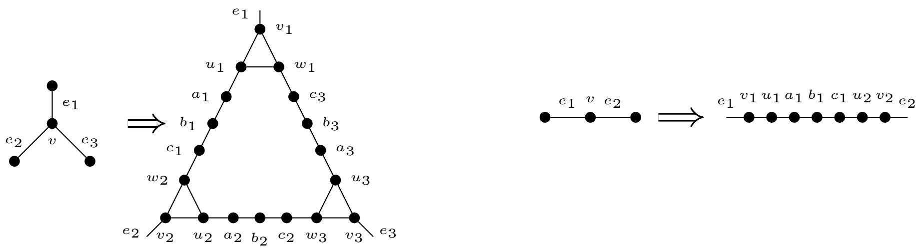
▷ Observation 3. For any $x \in X$, if $|D \cap V(G_x)| = 3$ then either $\{u^1_x, u^2_x, u^3_x\} \subset D$, $\{T^1_x, T^2_x, T^3_x\} \subset D$ or $\{F^1_x, F^2_x, F^3_x\} \subset D$.

Now, consider a clause $c \in C$ containing variables x_1, x_2 and x_3 and suppose without loss of generality that $T^1_{x_1}$ is adjacent to c (note that then the variable vertex x_1 is adjacent to $F^1_{x_1}$). If the clause vertex c belongs to D then, since D is efficient, $T^1_{x_1} \notin D$ and $u^1_{x_1}, F^1_{x_1} \notin D$ ($T^1_{x_1}$ would otherwise be dominated by at least two vertices) which contradicts Observation 3. Thus, no clause vertex belongs to D. Similarly, suppose that there exists $i \in \{1, 2, 3\}$ such that $x_i \in D$, say $x_1 \in D$ without loss of generality. Then, since D is efficient, $F^1_{x_1} \notin D$ and $T^1_{x_1}, u^2_{x_1} \notin D$ ($F^1_{x_1}$ would otherwise be dominated by at least two vertices) which again contradicts Observation 3. Thus, no variable vertex belongs to D. Finally, since D is efficient, $|D \cap V(K_c)| \leq 1$ and so, $|D \cap V(G_c)| = 1$ by Observation 1. ◁

Now by combining Claims 1 and 2, we obtain that Φ is satisfiable if and only if every minimum dominating set of G_Φ is efficient, that is, G_Φ is a Yes-instance for All Efficient MD. ◀

▶ **Theorem 7.** All Independent MD *is* NP*-hard when restricted to subcubic claw-free graphs.*

Proof. We give a reduction from Positive Exactly 3-Bounded 1-In-3 3-Sat, where each variable appears in exactly three clauses and only positively, each clause contains three positive literals, and we want a truth assignment such that each clause contains exactly one true literal. This problem is shown to be NP-complete in [14]. Given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Independent MD as follows. Consider the graph $G_\Phi = (V, E)$ constructed in the proof of Lemma 6 and let $V_i = \{v \in V : d_{G_\Phi}(v) = i\}$ for $i = 2, 3$ (note that no vertex in G_Φ has degree one). Then, for any $v \in V_3$, we replace the vertex v by the gadget G_v depicted in Fig. 3a; and for any $v \in V_2$, we replace the vertex v by the gadget G_v depicted in Fig. 3b. We denote by G'_Φ the resulting graph. Note that G'_Φ is claw-free and $\Delta(G'_\Phi) = 3$ (also note that no vertex in G'_Φ has degree one). It is shown in the proof of Lemma 6 that Φ is satisfiable if and only if G_Φ is a Yes-instance for All Efficient MD; we here show that G_Φ is a Yes-instance for All Efficient MD if and only if G'_Φ is a Yes-instance for All Independent MD. To this end, we first prove the following.



(a) $d_{G_\Phi}(v) = 3$. **(b)** $d_{G_\Phi}(v) = 2$.

Figure 3 The gadget G_v.

▷ Claim 3. $\gamma(G'_\Phi) = \gamma(G_\Phi) + 5|V_3| + 2|V_2|$.

Proof. Let D be a minimum dominating set of G_Φ. We construct a dominating set D' of G'_Φ as follows. For any $v \in D$, if $v \in V_3$, add v_1, v_2, v_3, b_1, b_2 and b_3 to D'; otherwise, add v_1, v_2 and b_1 to D'. For any $v \in V \setminus D$, let $u \in D$ be a neighbor of v, say $e_1 = uv$ without loss of generality. Then, if $v \in V_3$, add a_1, c_3, w_2, u_3 and b_2 to D'; otherwise, add a_1 and u_2 to D'. Clearly, D' is dominating and $|D'| = \gamma(G_\Phi) + 5|V_3| + 2|V_2| \geq \gamma(G'_\Phi)$.

▷ Observation 4. For any dominating set D' of G'_Φ, the following holds.

(i) For any $v \in V_2$, $|D' \cap V(G_v)| \geq 2$. Moreover, if equality holds then $D' \cap \{v_1, v_2\} = \emptyset$ and there exists $j \in \{1, 2\}$ such that $u_j \notin D'$.

(ii) For any $v \in V_3$, $|D' \cap V(G_v)| \geq 5$. Moreover, if equality holds then $D' \cap \{v_1, v_2, v_3\} = \emptyset$ and there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$.

(i) Clearly, $D' \cap \{v_1, u_1, a_1\} \neq \emptyset$ and $D' \cap \{c_1, u_2, v_2\} \neq \emptyset$ as u_1 and u_2 must be dominated. Thus, $|D' \cap V(G_v)| \geq 2$. Now, suppose that $D' \cap \{v_1, v_2\} \neq \emptyset$ say $v_1 \in D'$ without loss of generality. Then $D' \cap \{u_1, a_1, b_1\} \neq \emptyset$ as a_1 must be dominated which implies that $|D' \cap V(G_v)| \geq 3$ (recall that $D' \cap \{c_1, u_2, v_2\} \neq \emptyset$). Similarly, if both u_1 and u_2 belong to D', then $|D' \cap V(G_v)| \geq 3$ as $D' \cap \{a_1, b_1, c_1\} \neq \emptyset$ (b_1 would otherwise not be dominated).

(ii) Clearly, for any $i \in \{1, 2, 3\}$, $D' \cap \{a_i, b_i, c_i\} \neq \emptyset$ as b_i must be dominated. Now, if there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality, then $a_1, c_3 \in D'$ (one of u_1 and w_1 would otherwise not be dominated). But then, $D' \cap \{b_1, c_1, w_2\} \neq \emptyset$ as c_1 must be dominated, and $D' \cap \{a_3, b_3, u_3\} \neq \emptyset$ as a_3 must be dominated; and so, $|D' \cap V(G_v)| \geq 5$ (recall that $D' \cap \{a_2, b_2, c_2\} \neq \emptyset$). Otherwise, for any $j \in \{1, 2, 3\}$, $D' \cap \{u_j, v_j, w_j\} \neq \emptyset$ which implies that $|D' \cap V(G_v)| \geq 6$.

Now suppose that $D' \cap \{v_1, v_2, v_3\} \neq \emptyset$, say $v_1 \in D'$ without loss of generality. If there exists $j \neq 1$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 2$ without loss of generality, then $c_1, a_2 \in D'$ (one of u_2 and w_2 would otherwise not be dominated). But then, $D' \cap \{a_1, b_1, u_1\} \neq \emptyset$ as a_1 should be dominated, and $D' \cap \{b_2, c_2, w_3\} \neq \emptyset$ as c_2 must be dominated. Since $D' \cap \{a_3, b_3, c_3\} \neq \emptyset$, it then follows that $|D' \cap V(G_v)| \geq 6$. Otherwise, $D' \cap \{u_j, v_j, w_j\} \neq \emptyset$ for any $j \in \{1, 2, 3\}$ and so, $|D' \cap V(G_v)| \geq 6$ (recall that $D' \cap \{a_i, b_i, c_i\} \neq \emptyset$ for any $i \in \{1, 2, 3\}$). ⌟

▷ Observation 5. If D' is a minimum dominating set of G'_Φ, then $|D' \cap V(G_v)| \leq 3$ for any $v \in V_2$ and $|D' \cap V(G_v)| \leq 6$ for any $v \in V_3$.

Indeed, if $v \in V_2$ then $\{v_1, b_1, v_2\}$ is a dominating set of $V(G_v)$; and if $v \in V_3$, then $\{v_1, v_2, v_3, b_1, b_2, b_3\}$ is a dominating set of $V(G_v)$. ⌟

Now, consider a minimum dominating set D' of G'_Φ and let $D_3 = \{v \in V_3 : |D' \cap V(G_v)| = 6\}$ and $D_2 = \{v \in V_2 : |D' \cap V(G_v)| = 3\}$. We claim that $D = D_3 \cup D_2$ is a dominating set of G_Φ. Indeed, consider a vertex $v \in V \setminus D$. We distinguish two cases depending on whether $v \in V_2$ of $v \in V_3$.

Case 1. $v \in V_2$. Then $|D' \cap V(G_v)| = 2$ by construction, which by Observation 4(i) implies that there exists $j \in \{1, 2\}$ such that $D' \cap \{v_j, u_j\} = \emptyset$, say $j = 1$ without loss of generality. Since v_1 must be dominated, v_1 must then have a neighbor x_i belonging to D', for some vertex x adjacent to v in G_Φ. But then, it follows from Observation 4 that $|D' \cap V(G_x)| > 2$ if $x \in V_2$, and $|D' \cap V(G_x)| > 5$ if $x \in V_3$ (indeed, $x_i \in D'$); thus, $x \in D$.

Case 2. $v \in V_3$. Then $|D' \cap V(G_v)| = 5$ by construction, which by Observation 4(ii) implies that there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality. Since v_1 must be dominated, v_1 must then have a neighbor x_i belonging to D', for some vertex x adjacent to v in G_Φ. But then, it follows from Observation 4 that $|D' \cap V(G_x)| > 2$ if $x \in V_2$, and $|D' \cap V(G_x)| > 5$ if $x \in V_3$ (indeed, $x_i \in D'$); thus, $x \in D$.

Hence, D is a dominating set of G_Φ. Moreover, it follows from Observations 4 and 5 that $|D'| = 6|D_3| + 5|V_3 \setminus D_3| + 3|D_2| + 2|V_2 \setminus D_2| = |D| + 5|V_3| + 2|V_2|$. Thus, $\gamma(G'_\Phi) = |D'| \geq \gamma(G_\Phi) + 5|V_3| + 2|V_2|$ and so, $\gamma(G'_\Phi) = \gamma(G_\Phi) + 5|V_3| + 2|V_2|$. Finally note that this implies that the constructed dominated set D is in fact minimum. ◁

We next show that G_Φ is a Yes-instance for All Efficient MD if and only if G'_Φ is a Yes-instance for All Independent MD. Since Φ is satisfiable if and only if G_Φ is a Yes-instance for All Efficient MD, as shown in the proof of Lemma 6, this would conclude the proof.

Assume first that G_Φ is a Yes-instance for All Efficient MD and suppose to the contrary that G'_Φ is a No-instance for All Independent MD that is, G'_Φ has a minimum dominating set D' which is not independent. Denote by D the minimum dominating set of G_Φ constructed from D' according to the proof of Claim 3. Let us show that D is not efficient. Consider two adjacent vertices $a, b \in D'$. If a and b belong to gadgets G_x and G_v respectively, for two adjacent vertices x and v in G_Φ, that is, a is of the form x_i and b is of the form v_j, then by Observation 4 $x, v \in D$ and so, D is not efficient. Thus, it must be that a and b both belong the same gadget G_v, for some $v \in V_2 \cup V_3$. We distinguish cases depending on whether $v \in V_2$ or $v \in V_3$.

Case 1. $v \in V_2$. Suppose that $|D' \cap V(G_v)| = 2$. Then by Observation 4(i), $D' \cap \{v_1, v_2\} = \emptyset$ and there exists $j \in \{1, 2\}$ such that $u_j \notin D'$, say $u_1 \notin D'$ without loss of generality. Then, necessarily $a_1 \in D'$ (u_1 would otherwise not be dominated) and so, $b_1 \in D'$ as $D' \cap V(G_v)$ contains an edge and $|D' \cap V(G_v)| = 2$ by assumption; but then, u_2 is not dominated. Thus, $|D' \cap V(G_v)| \geq 3$ and we conclude by Observation 5 that in fact, equality holds. Note that consequently, $v \in D$. We claim that then, $|D' \cap \{v_1, v_2\}| \leq 1$. Indeed, if both v_1 and v_2 belong to D', then $b_1 \in D'$ (since $|D' \cap V(G_v)| = 3$, D' would otherwise not be dominating) which contradicts that fact that $D' \cap V(G_v)$ contains an edge. Thus, $|D' \cap \{v_1, v_2\}| \leq 1$ and we may assume without loss of generality that $v_2 \notin D'$. Let $x_i \neq u_2$ be the other neighbor of v_2 in G'_Φ, where x is a neigbhor of v in G_Φ.
Suppose first that $x \in V_2$. Then, $|D' \cap V(G_x)| = 2$ for otherwise x would belong to D and so, D would contain the edge vx. It then follows from Observation 4(i) that there exists $j \in \{1, 2\}$ such that $D' \cap \{x_j, y_j\} = \emptyset$, where y_j is the neighbor of x_j in $V(G_x)$.

We claim that $j \neq i$; indeed, if $j = i$, since $v_2, x_i, y_i \notin D'$, x_i would not be dominated. But then, x_j must have a neighbor $t_k \neq y_j$, for some vertex t adjacent to x in G_Φ, which belongs to D'; it then follows from Observation 4 and the construction of D that $t \in D$ and so, x has two neighbors in D, namely v and t, a contradiction.
Second, suppose that $x \in V_3$. Then, $|D' \cap V(G_x)| = 5$ for otherwise x would belong to D and so, D would contain the edge vx. It then follows from Observation 4(ii) that there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{x_j, y_j, z_j\} = \emptyset$, where y_j and z_j are the two neighbors of x_j in $V(G_x)$. We claim that $j \neq i$; indeed, if $j = i$, since $v_2, x_i, y_i, z_i \notin D'$, x_i would not be dominated. But then, x_j must have a neighbor $t_k \neq y_j, z_j$, for some vertex t adjacent to x in G_Φ, which belongs to D'; it then follows from Observation 4 and the construction of D that $t \in D$ and so, x has two neighbors in D, namely v and t, a contradiction.

Case 2. $v \in V_3$. Suppose that $|D' \cap V(G_v)| = 5$. Then, by Observation 4(ii), $D' \cap \{v_1, v_2, v_3\} = \emptyset$ and there exists $j \in \{1, 2, 3\}$ such that $D' \cap \{u_j, v_j, w_j\} = \emptyset$, say $j = 1$ without loss of generality. Then, $a_1, c_3 \in D'$ (one of u_1 and w_1 would otherwise not be dominated), $D' \cap \{c_1, w_2, u_2\} \neq \emptyset$ (w_2 would otherwise not be dominated), $D' \cap \{a_3, u_3, w_3\} \neq \emptyset$ (u_3 would otherwise not be dominated) and $D' \cap \{a_2, b_2, c_2\} \neq \emptyset$ (b_2 would otherwise not be dominated); in particular, $b_1, b_3 \notin D'$ as $|D' \cap V(G_v)| = 5$ by assumption. Since $D' \cap V(G_v)$ contains an edge, it follows that either $u_2, a_2 \in D'$ or $c_2, w_3 \in D'$; but then, either c_1 or a_3 is not dominated, a contradiction. Thus, $|D' \cap V(G_v)| \geq 6$ and we conclude by Observation 5 that in fact, equality holds. Note that consequently, $v \in D$. It follows that $\{v_1, v_2, v_3\} \not\subset D'$ for otherwise $D' \cap V(G_v) = \{v_1, v_2, v_3, b_1, b_2, b_3\}$ and so, $D' \cap V(G_v)$ contains no edge. Thus, we may asssume without loss of generality that $v_1 \notin D'$. Denoting by $x_i \neq u_1, w_1$ the third neighbor of v_1, where x is a neighbor of v in G_Φ, we then proceed as in the previous case to conclude that x has two neighbors in D.

Thus, D is not efficient, which contradicts the fact that G_Φ is a Yes-instance for All Efficient MD. Hence, every minimum dominating set of G'_Φ is independent i.e., G'_Φ is a Yes-instance for All Independent MD.

Conversely, assume that G'_Φ is a Yes-instance for All Independent MD and suppose to the contrary that G_Φ is a No-instance for All Efficient MD that is, G_Φ has a minimum dominating set D which is not efficient. Let us show that D either contains an edge or can be transformed into a minimum dominating set of G_Φ containing an edge. Since any minimum dominating of G'_Φ constructed according to the proof of Claim 3 from a minimum dominating set of G_Φ containing an edge, also contains an edge, this would lead to a contradiction and thus conclude the proof.

Suppose that D contains no edge. Since D is not efficient, there must then exist a vertex $v \in V \setminus D$ such that v has two neighbors in D. We distinguish cases depending on which type of vertex v is.

Case 1. *v is a variable vertex.* Suppose that $v = x_1$ in some clause gadget G_c, where $c \in C$ contains variables x_1, x_2 and x_3, and assume without loss of generality that x_1 is adjacent to $F^1_{x_1}$. By assumption, $F^1_{x_1}, l_{\{x_1\}} \in D$ which implies that $D \cap \{l_{\{x_2\}}, l_{\{x_3\}}, T^1_{x_1}, u^2_{x_1}\} = \emptyset$ (D would otherwise contain an edge). We may then assume that $F^i_{x_2}$ and $F^j_{x_3}$, where $F^i_{x_2}x_2, F^j_{x_3}x_3 \in E(G_\Phi)$, belong to D; indeed, since x_2 (resp. x_3) must be dominated, $D \cap \{F^i_{x_2}, x_2\} \neq \emptyset$ (resp. $D \cap \{F^j_{x_3}, x_3\} \neq \emptyset$) and since $l_{\{x_1\}} \in D$, $(D \setminus \{x_2\}) \cup \{F^i_{x_2}\}$ (resp. $(D \setminus \{x_3\}) \cup \{F^j_{x_3}\}$) remains dominating. We may then assume that $T^i_{x_2}, T^j_{x_3} \notin D$ for otherwise D would contain an edge. It follows that $c \in D$ (c would otherwise not be dominated); but then, it suffices to consider $(D \setminus \{c\}) \cup \{T^1_{x_1}\}$ to obtain a minimum dominating set of G_Φ containing an edge.

Case 2. *$v = u_x^i$ for some variable $x \in X$ and $i \in \{1,2,3\}$.* Assume without loss of generality that $i = 1$. Then $T_x^1, F_x^3 \in D$ by assumption, which implies that $F_x^1, T_x^3 \notin D$ (D would otherwise contain an edge). But then, $|D \cap \{u_x^2, F_x^2, T_x^2, u_x^3\}| \geq 2$ as u_x^2 and u_x^3 must be dominated; and so, $(D \setminus \{u_x^3, F_x^2, T_x^2, u_x^2\}) \cup \{F_x^2, T_x^2\}$ is a dominating set of G_Φ of size at most that of D which contains an edge.

Case 3. *v is a clause vertex.* Suppose that $v = c$ for some clause $c \in C$ containing variables x_1, x_2 and x_3, and assume without loss of generality that c is adjacent to $T_{x_i}^1$ for any $i \in \{1,2,3\}$. By assumption c has two neighbors in D, say $T_{x_1}^1$ and $T_{x_2}^1$ without loss of generality. Since D contains no edge, it follows that $F_{x_1}^1, F_{x_2}^1 \notin D$; but then, $|D \cap \{x_1, x_2, l_{\{x_1\}}, l_{\{x_2\}}\}| \geq 2$ (one of x_1 and x_2 would otherwise not be dominated) and so, $(D \setminus \{x_1, x_2, l_{\{x_1\}}, l_{\{x_2\}}\}) \cup \{l_{\{x_1\}}, l_{\{x_2\}}\}$ is a dominating set of G_Φ of size at most that of D which contains an edge.

Case 4. *$v \in V(K_c)$ for some clause $c \in C$.* Denote by x_1, x_2 and x_3 the variables contained in c and assume without loss of generality that $v = l_{\{x_1\}}$. Since $l_{\{x_1\}}$ has two neighbors in D and D contains no edge, necessarily $x_1 \in D$. Now assume without loss of generality that x_1 is adjacent to $F_{x_1}^1$ (note that by construction, c is then adjacent to $T_{x_1}^1$). Then, $F_{x_1}^1 \notin D$ (D would otherwise contain an edge) and $T_{x_1}^1, u_{x_1}^2 \notin D$ for otherwise $(D \setminus \{x_1\}) \cup \{F_{x_1}^1\}$ would be a minimum dominating set of G_Φ containing an edge (recall that by assumption, $D \cap V(K_c) \neq \emptyset$). It follows that $T_{x_1}^2 \in D$ ($u_{x_1}^2$ would otherwise not be dominated) and so, $F_{x_1}^2 \notin D$ as D contains no edge. It follows that $|D \cap \{u_{x_1}^1, F_{x_1}^3, T_{x_1}^3, u_{x_1}^3\}| \geq 2$ as $u_{x_1}^1$ and $u_{x_1}^3$ must be dominated. Now if c belongs to D, then $(D \setminus \{u_{x_1}^1, F_{x_1}^3, T_{x_1}^3, u_{x_1}^3\}) \cup \{F_{x_1}^3, T_{x_1}^3\}$ is a dominating set of G_Φ of size at most that of D which contains an edge. Thus, we may assume that $c \notin D$ which implies that $u_{x_1}^1 \in D$ ($T_{x_1}^1$ would otherwise not be dominated) and that there exists $j \in \{2,3\}$ such that $T_{x_j}^i \in D$ with $cT_{x_j}^i \in E(G_\Phi)$ (c would otherwise not be dominated). Now, since $u_{x_1}^3$ must be dominated and $F_{x_1}^2 \notin D$, it follows that $D \cap \{u_{x_1}^3, T_{x_1}^3\} \neq \emptyset$ and we may assume that in fact $T_{x_1}^3 \in D$ (recall that $T_{x_1}^2 \in D$ and so, $F_{x_1}^2$ is dominated). But then, by considering the minimum dominating set $(D \setminus \{u_{x_1}^1\}) \cup \{T_{x_1}^1\}$, we fall back into Case 3 as c is then dominated by both $T_{x_1}^1$ and $T_{x_j}^i$.

Case 5. *v is a true vertex.* Assume without loss of generality that $v = T_x^1$ for some variable $x \in X$. Suppose first that $u_x^1 \in D$. Then since D contains no edge, $F_x^3 \notin D$; furthermore, denoting by $t \neq u_x^1, T_x^3$ the variable vertex adjacent to F_x^3, we also have $t \notin D$ for otherwise $(D \setminus \{u_x^1\}) \cup \{F_x^3\}$ would be a minimum dominating set containing an edge (recall that T_x^1 has two neighbors in D by assumption). But then, since t must be dominated, it follows that the second neighbor of t must belong to D; and so, by considering the minimum dominating set $(D \setminus \{u_x^1\}) \cup \{F_x^3\}$, we fall back into Case 1 as the variable vertex t is then dominated by two vertices. Thus, we may assume that $u_x^1 \notin D$ which implies that $F_x^1, c \in D$, where c is the clause vertex adjacent to T_x^1. Now, denote by $x_1 = x$, x_2 and x_3 the variables contained in c (note that by construction, x_1 is then adjacent to $F_{x_1}^1$). Then, $x_1 \notin D$ (D would otherwise contain the edge $F_{x_1}^1 x_1$) and we may assume that $l_{\{x_1\}} \notin D$ (we otherwise fall back into Case 1 as x_1 would then have two neighbors in D). It follows that $D \cap V(K_c) \neq \emptyset$ ($l_{\{x_1\}}$ would otherwise not be dominated) and since D contains no edge, in fact $|D \cap V(K_c)| = 1$, say $l_{\{x_2\}} \in D$ without loss of generality. Then, $x_2 \notin D$ as D contains no edge and we may assume that $F_{x_2}^j \notin D$, where $F_{x_2}^j$ is the false vertex adjacent to x_2, for otherwise we fall back into Case 1. In the following, we assume without loss of generality that $j = 1$, that is, x_2 is adjacent to $F_{x_2}^1$ (note that by construction, c is then adjacent to $T_{x_2}^1$). Now, since the clause vertex c belongs to D by assumption, it follows that $T_{x_2}^1 \notin D$ (D would otherwise contain the edge $cT_{x_2}^1$); and as shown previously, we may assume that $u_{x_2}^1 \notin D$ (indeed, $T_{x_2}^1$ would otherwise have two neighbors in D, namely c and $u_{x_2}^1$, but this case has already been dealt with). Then, since $u^1_{x_2}$ and $F^1_{x_2}$ must be dominated, necessarily $F^3_{x_2}$ and $u^2_{x_2}$ belong to D (recall that $D \cap \{x_2, F^1_{x_2}, T^1_{x_2}, u^1_{x_2}\} = \emptyset$) which implies that $T^3_{x_2}, T^2_{x_2} \notin D$ (D would otherwise contain an edge). Now since $u^3_{x_2}$ must be dominated, $D \cap \{u^3_{x_2}, F^2_{x_2}\} \neq \emptyset$ and we may assume without loss of generality that in fact, $F^2_{x_2} \in D$. But then, by considering the minimum dominating set $(D \setminus \{u^2_{x_2}\}) \cup \{F^1_{x_2}\}$, we fall back into Case 1 as x_2 is then dominated by two vertices.

Case 6. *v is a false vertex.* Assume without loss of generality that $v = F^1_{x_1}$ for some variable $x_1 \in X$ and let $c \in C$ be the clause whose corresponding clause vertex is adjacent to $T^1_{x_1}$. Denote by x_2 and x_3 the two other variables contained in c. Suppose first that $x_1 \in D$. Then, we may assume that $D \cap V(K_c) = \emptyset$ for otherwise either D contains an edge (if $l_{\{x_1\}} \in D$) or we fall back into Case 4 ($l_{\{x_1\}}$ would indeed have two neighbors in D). Since every vertex of K_c must be dominated, it then follows that $x_2, x_3 \in D$; but then, by considering the minimum dominating set $(D \setminus \{x_1\}) \cup \{l_{\{x_1\}}\}$ (recall that $F^1_{x_1}$ has two neighbors in D by assumption), we fall back into Case 4 as $l_{\{x_2\}}$ is then dominated by two vertices. Thus, we may assume that $x_1 \notin D$ which implies that $T^1_{x_1}, u^2_{x_1} \in D$ and $T^2_{x_1}, u^1_{x_1} \notin D$ as D contains no edge. Now, denote by c' the clause vertex adjacent to $T^2_{x_1}$. Then, we may assume that $c' \notin D$ for otherwise we fall back into Case 5 ($T^2_{x_1}$ would indeed have two neighbors in D); but then, there must exist a true vertex, different from $T^2_{x_1}$, adjacent to c' and belonging to D (c' would otherwise not be dominated) and by considering the minimum dominating set $(D \setminus \{u^2_{x_1}\}) \cup \{T^2_{x_1}\}$, we then fall back into Case 3 (c' would indeed be dominated by two vertices).

Consequently, G_Φ has a minimum dominating set which is not independent which implies that G'_Φ also has a minimum dominating set which is not independent, a contradiction which concludes the proof. ◀

Theorem 2 now easily follows from Fact 1 and Theorem 7.

5 The proof of Theorem 3

In this section, we show that 1-Edge Contraction(γ) is coNP-hard when restricted to $2P_3$-free graphs. To this end, we prove the following.

▶ **Theorem 8.** All Independent MD *is* NP*-hard when restricted to* $2P_3$*-free graphs.*

Proof. We reduce from 3-Sat: given an instance Φ of this problem, with variable set X and clause set C, we construct an equivalent instance of All Independent MD as follows. For any variable $x \in X$, we introduce a copy of C_3, which we denote by G_x, with one distinguished *positive literal vertex* x and one distinguished *negative literal vertex* $\bar{x}$; in the following, we denote by u_x the third vertex in G_x. For any clause $c \in C$, we introduce a *clause vertex* c; we then add an edge between c and the (positive or negative) literal vertices whose corresponding literal occurs in c. Finally, we add an edge between any two clause vertices so that the set of clause vertices induces a clique denoted by K in the following. We denote by G_Φ the resulting graph.

▷ Observation 1. For any dominating set D of G_Φ and any variable $x \in X$, $|D \cap V(G_x)| \geq 1$. In particular, $\gamma(G_\Phi) \geq |X|$.

▷ Claim 1. Φ is satisfiable if and only if $\gamma(G_\Phi) = |X|$.

Proof. Assume that Φ is satisfiable and consider a truth assignment satisfying Φ. We construct a dominating set D of G_Φ as follows. For any variable $x \in X$, if x is true, add the positive literal vertex x to D; otherwise, add the negative variable vertex $\bar{x}$ to D. Clearly, D is dominating and we conclude by Observation 1 that $\gamma(G_\Phi) = |X|$.

Conversely, assume that $\gamma(G_\Phi) = |X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 1, $|D \cap V(G_x)| = 1$ for any $x \in X$. It follows that $D \cap K = \emptyset$ and so, every clause vertex must be adjacent to some (positive or negative) literal vertex belonging to D. We thus construct a truth assignment satisfying Φ as follows: for any variable $x \in X$, if the positive literal vertex x belongs to D, set x to true; otherwise, set x to false. ◁

▷ Claim 2. $\gamma(G_\Phi) = |X|$ if and only if every minimum dominating set of G_Φ is independent.

Proof. Assume that $\gamma(G_\Phi) = |X|$ and consider a minimum dominating set D of G_Φ. Then by Observation 1, $|D \cap V(G_x)| = 1$ for any $x \in X$. It follows that $D \cap K = \emptyset$ and since $N[V(G_x)] \cap N[V(G_{x'})] \subset K$ for any two $x, x' \in X$, D is independent.

Conversely, consider a minimum dominating set D of G_Φ. Since D is independent, $|D \cap V(G_x)| \leq 1$ for any $x \in X$ and we conclude by Observation 1 that in fact, equality holds. Now suppose that there exists $c \in C$, containing variables x_1, x_2 and x_3, such that the corresponding clause vertex c belongs to D (note that since D is independent, $|D \cap K| \leq 1$). Assume without loss of generality that x_1 occurs positively in c, that is, c is adjacent to the positive literal vertex x_1. Then, $x_1 \notin D$ since D is independent and so, either $u_{x_1} \in D$ or $\bar{x_1} \in D$. In the first case, we immediately obtain that $(D \setminus \{u_{x_1}\}) \cup \{x_1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. In the second case, since $c \in D$, any vertex dominated by $\bar{x_1}$ is also dominated by c; thus, $(D \setminus \{\bar{x_1}\}) \cup \{x_1\}$ is a minimum dominating set of G_Φ containing an edge, a contradiction. Consequently, $D \cap K = \emptyset$ and so, $\gamma(G_\Phi) = |D| = |X|$. ◁

Now by combining Claims 1 and 2, we obtain that Φ is satisfiable if and only if every minimum dominating set of G_Φ is independent, that is, G_Φ is a Yes-instance for All Independent MD. There remains to show that G_Φ is $2P_3$-free. To see this, it suffices to observe that any induced P_3 of G_Φ contains at least one vertex in the clique K. This concludes the proof. ◀

Theorem 3 now easily follows from Fact 1 and Theorem 8.

6 Conclusion

In this work, we establish a complexity dichotomy for 1-Edge Contraction(γ) on H-free graphs when H is a connected graph. If we do not require H to be connected, there only remains to settle the complexity status of 1-Edge Contraction(γ) restricted to H-free graphs when $H = P_3 + qP_2 + pK_1$, with $q \geq 1$ and $p \geq 0$.

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