On Affine Reachability Problems

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Abstract

We analyze affine reachability problems in dimensions 1 and 2. We show that the reachability problem for 1-register machines over the integers with affine updates is PSPACE-hard, hence PSPACE-complete, strengthening a result by Finkel et al. that required polynomial updates. Building on recent results on two-dimensional integer matrices, we prove NP-completeness of the mortality problem for 2-dimensional integer matrices with determinants +1 and 0. Motivated by tight connections with 1-dimensional affine reachability problems without control states, we also study the complexity of a number of reachability problems in finitely generated semigroups of 2-dimensional upper-triangular integer matrices.

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1 Introduction

Counter machines. *Counter machines* are abstract models of computation, consisting of a finite control and a set of registers which store numbers. Upon taking a transition, the machine may perform simple arithmetic on the registers. There is a great variety of such machines, depending on the domain of the registers ($\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, ...), whether the content of the registers influences the control flow (e.g., via zero tests), and the types of allowed register updates (changes can only be additive, linear, affine, polynomial, ...).

As a model of programs with arithmetic, counter machines relate to program analysis and verification. They also provide natural classes of finitely presented systems with infinitely many states, with a regular-shaped transition structure. Minsky [18] showed that counter machines with nonnegative integer registers, additive updates, and zero tests are Turing-powerful. *Vector addition systems with states (VASS)*, which are roughly equivalent to *Petri nets*, are a related well-studied model without zero tests. Reachability in this model is decidable, albeit with very high complexity [10]. Recent work [4] considers reachability in certain variants of VASS, including in affine VASS, which are closely related to affine register machines (see below) but have multiple counters.

In this paper, **we establish the precise complexity of reachability in *affine register machines*.** These are counter machines with a single integer register (two registers already lead to undecidability [26, Chapter 2.5]), no zero tests, and affine register updates; i.e., the transitions are labelled with updates of the form $x := ax + b$, where x stands for the register and a, b are integer coefficients. Finkel et al. [12] considered a more general model, *polynomial register machines*, with the difference that the updates consist of arbitrary integer polynomials, not just affine polynomials $ax + b$. The main result of [12] is that reachability

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in polynomial register machines is PSPACE-complete. We show that reachability in affine register machines is also PSPACE-hard, hence PSPACE-complete. Niskanen [20] strengthened the lower bound from [12] in an orthogonal direction, by showing PSPACE-hardness in the case with polynomial updates but without control states.

As we explain in the following (see also Proposition 1 below), the stateless case is intimately connected with finitely generated monoids over two-dimensional upper-triangular integer matrices. This leads us to investigate several natural reachability problems in such monoids.

Matrix monoids. A finite set of matrices $\mathcal{M} \subset \mathbb{Q}^{d\times d}$ generates a monoid $\langle\mathcal{M}\rangle$ under matrix multiplication, i.e., $\langle\mathcal{M}\rangle$ is the set of products of matrices from $\mathcal{M}$, including the identity matrix, which we view as the empty product. Algorithmic problems about such monoids are hard, often undecidable. For example, Paterson [23] showed in 1970 that the *mortality* problem, i.e., deciding whether the zero matrix is in the generated monoid, is undecidable, even for integer matrices with $d = 3$. It remains undecidable for $d = 3$ with $|\mathcal{M}| = 6$ and for $d = 18$ with $|\mathcal{M}| = 2$; see [19]. Mortality for two-dimensional matrices is known to be NP-hard [1], but decidability remains a long-standing open problem; see, e.g., [25].

Mortality is a special case of the *membership* problem: given $\mathcal{M}$ and a matrix T, is $T \in \langle\mathcal{M}\rangle$? Two other natural problems consider certain linear projections of the matrices in $\langle\mathcal{M}\rangle$: The *vector reachability* problem asks, given $\mathcal{M}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{Q}^d$, if there is a matrix $M \in \langle\mathcal{M}\rangle$ such that $M\vec{x} = \vec{y}$. Similarly, the *scalar reachability* problem asks if there is a matrix $M \in \langle\mathcal{M}\rangle$ such that $\vec{y}^T M\vec{x} = \lambda$ holds for given vectors $\vec{x}, \vec{y}$ and a given scalar $\lambda \in \mathbb{Q}$. For $d = 2$ none of these problems are known to be decidable, not even for integer matrices. In the case $d = 2$, mortality *is* known to be decidable when $|\mathcal{M}| = 2$ holds [6], and for integer matrices whose determinants are in $\{-1, 0, +1\}$ (see [21]).

Even the case of a single matrix, i.e., $|\mathcal{M}| = 1$, is very difficult; see [22] for a survey. This case is closely related to the algorithmic analysis of *linear recurrence sequences*, which are sequences $u_0, u_1, \ldots$ of numbers such that there are constants $a_1, \ldots, a_d$ such that $u_{n+d} = a_1u_{n+d-1} + a_2u_{n+d-2} + \cdots + a_du_n$ holds for all $n \in \mathbb{N}$. In the case $|\mathcal{M}| = 1$ the vector reachability problem is referred to as the *orbit* problem, and the scalar reachability problem as the *Skolem* problem. The orbit problem is decidable in polynomial time [16], but the Skolem problem is only known to be decidable for $d \leq 4$ (this requires Baker's Theorem) [22, 9].

In the following, we do not restrict $|\mathcal{M}|$ but focus on integer matrices in $d = 2$. Recently, there has been steady progress for certain special cases. It was shown by Potapov and Semukhin [25] that the membership problem for two-dimensional integer matrices is decidable for non-singular matrices. This result builds on automata-theoretic techniques developed, e.g., in [8], where it was shown that the problem of deciding whether $\langle\mathcal{M}\rangle$ is a group is decidable. At its heart, this technique exploits the special structure of the group of matrices with determinants ± 1 and its subgroups. For matrices with determinant 1, further results are known, namely decidability of vector reachability [24] and NP-completeness of the membership problem [2]. If all matrices in $\mathcal{M}$ have determinant 1 and $\mathcal{M}$ is closed under inverses, then $\langle\mathcal{M}\rangle$ is a group. In this case, one can decide in polynomial time for a given matrix M whether M or $-M$ is in $\langle\mathcal{M}\rangle$ [14].

Building on these recent results [24, 2] **we prove that the mortality problem for two-dimensional integer matrices with determinants +1 or 0 is NP-complete**. The main result of [1] was NP-hardness for the same problem but allowing also for determinant -1. Thus, we strengthen the lower bound from [1] by disallowing determinant -1, and our subset-sum based proof is considerably simpler.

We then focus on *upper-triangular* integer matrices, i.e., integer matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Curiously, decidability of the membership, vector reachability, and scalar reachability problems are still challenging, and indeed open, despite the severe restriction on the matrix shape and dimension. This class of reachability problems is motivated by its tight connection to (stateless) affine reachability. For instance, *affine reachability over* $\mathbb{Q}$ reduces to scalar reachability for upper-triangular two-dimensional integer matrices; see Proposition 1. Affine reachability over $\mathbb{Q}$ asks, given a set of affine rational functions in one variable and two rational numbers $x, y \in \mathbb{Q}$, whether x can be transformed into y using one or more applications of the given functions, chosen nondeterministically.

Whereas affine reachability over $\mathbb{Z}$ is in PSPACE by [12], decidability of affine reachability over $\mathbb{Q}$ is open. The problem is related to the reachability problem with a single, but only piecewise affine, function ("*piecewise affine maps*"); this problem is not known to be decidable either; see [17, 7]. Variants of piecewise affine reachability, also over $\mathbb{Z}$, are studied in [3].

Organization of the paper. In Section 2 we state tight (perhaps folklore) connections between (1) reachability problems in monoids over two-dimensional upper-triangular integer matrices, and (2) reachability problems of one-dimensional affine functions. We then make the following contributions:

1. In Section 3 we show that reachability in affine register machines is PSPACE-hard, hence PSPACE-complete.
2. In Section 4 we prove NP-completeness of the mortality problem for two-dimensional integer matrices with determinants $+1$ or 0.
3. In Section 5 we study the complexity reachability problems in monoids over two-dimensional upper-triangular integer matrices:
   a. In Section 5.1 we study the case with ± 1 on the diagonal. Establishing a connection with so-called $\mathbb{Z}$-*VASS* [15] allows us to prove NP-completeness, although we show that the case where all generator matrices have determinant -1 is in P by a reduction to a linear system of Diophantine equations over the integers.
   b. In Section 5.2 we study vector reachability. We show that the problem is hard for affine reachability over $\mathbb{Q}$, hence decidability requires a breakthrough, but the case where the bottom-right entries are non-zero is in PSPACE.
   c. In Section 5.3 we study the membership problem. If both diagonal entries are non-zero, we show that the problem is NP-complete, which in turn shows NP-completeness of the following problem: given $n+1$ non-constant affine functions over $\mathbb{Z}$ in one variable, can the $n+1$st function be represented as a composition of the other n functions? The case where only one of the diagonal entries is restricted to be non-zero is decidable in PSPACE. Finally, for the case where both diagonal entries may be 0, we establish reductions between membership and scalar reachability, suggesting that decidability of membership would also require a breakthrough.

We conclude in Section 6. For space reasons, some missing proofs are only available in the full version of the paper.

2 Preliminaries

We write $\mathbb{Z}$ for the set of integer numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of nonnegative integers, and $\mathbb{Q}$ for the set of rationals. We write UT for the set (and the monoid under matrix multiplication) of two-dimensional upper-triangular integer matrices:

$$\mathsf{UT} := \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\}$$

We may drop the 0 in the bottom-left corner and write $\begin{pmatrix} x & y \\ & z \end{pmatrix}$ for matrices in UT. Let $\Phi(A)$ be a constraint for $A \in \mathsf{UT}$. We write $\mathsf{UT}[\Phi(A)] := \{A \in \mathsf{UT} \mid \Phi(A)\}$, e.g., $\mathsf{UT}[A_{22} = 1]$ denotes the set of all upper-triangular matrices whose bottom-right coefficient equals 1.

For a finite set $\mathcal{M}$ of matrices, we write $\langle \mathcal{M} \rangle$ for the monoid generated by $\mathcal{M}$ under matrix multiplication. In this paper we consider mainly the following reachability problems:

- **Membership:** Given a finite set $\mathcal{M} \subseteq \mathsf{UT}$, and a matrix $T \in \mathsf{UT}$, is $T \in \langle \mathcal{M} \rangle$?
- **Vector reachability:** Given a finite set $\mathcal{M} \subseteq \mathsf{UT}$, and vectors $\vec{x}, \vec{y} \in \mathbb{Z}^2$, is there a matrix $M \in \langle \mathcal{M} \rangle$ such that $M\vec{x} = \vec{y}$?
- **Scalar reachability:** Given a finite set $\mathcal{M} \subseteq \mathsf{UT}$, vectors $\vec{x}, \vec{y} \in \mathbb{Z}^2$, and a scalar $\lambda \in \mathbb{Z}$, is there a matrix $M \in \langle \mathcal{M} \rangle$ such that $\vec{y}^T M \vec{x} = \lambda$? We refer to the special case with $\lambda = 0$ as the 0-*reachability* problem.

We write $\mathsf{Aff}(\mathbb{Z})$ for the set (and the monoid under function composition) of affine functions:

$$\mathsf{Aff}(\mathbb{Z}) \;:=\; \{x \mapsto ax + b \mid a, b \in \mathbb{Z}\} \;\subseteq\; \mathbb{Z}^{\mathbb{Z}} \quad (\text{where } \mathbb{Z}^{\mathbb{Z}} = \{f : \mathbb{Z} \to \mathbb{Z}\})$$

Define $\mathsf{Aff}(\mathbb{Q})$ similarly, with $\mathbb{Z}$ replaced by $\mathbb{Q}$. For a finite set $\mathcal{A}$ of affine functions, we write $\langle \mathcal{A} \rangle$ for the monoid (i.e., including the identity function $x \mapsto x$) generated by $\mathcal{A}$ under function composition. The motivation to study the matrix reachability problems above is their relationship to the following one-dimensional affine reachability problems:

- **Affine membership over $\mathbb{Z}$:** Given a finite set $\mathcal{A} \subseteq \mathsf{Aff}(\mathbb{Z})$, and a function $f \in \mathsf{Aff}(\mathbb{Z})$, is $f \in \langle \mathcal{A} \rangle$?
- **Affine reachability over $\mathbb{Z}$:** Given a finite set $\mathcal{A} \subseteq \mathsf{Aff}(\mathbb{Z})$, and numbers $x, y \in \mathbb{Z}$, is there a function $f \in \langle \mathcal{A} \rangle$ such that $f(x) = y$?
- **Affine reachability over $\mathbb{Q}$:** The same problem with $\mathbb{Z}$ replaced by $\mathbb{Q}$.

These problems are linked by the following proposition. Recall from the definitions above that the matrices are restricted to be two-dimensional upper-triangular integer matrices.

▶ **Proposition 1.**
1. *Affine membership over $\mathbb{Z}$ is logspace inter-reducible with (matrix) membership restricted to matrices with* 1 *in the bottom-right corner.*
2. *Affine reachability over $\mathbb{Z}$ is logspace inter-reducible with vector reachability restricted to matrices with* 1 *in the bottom-right corner and vectors with* 1 *in the bottom entry.*
3. *Affine reachability over $\mathbb{Q}$ is logspace inter-reducible with* 0*-reachability restricted to matrices with non-zero entries in the bottom-right corner and vectors $\vec{x}, \vec{y} \in \mathbb{Z}^2$ such that the bottom entry of $\vec{x}$ and the top entry of $\vec{y}$ are non-zero.*

Proof. The proof is fairly standard and can be found in the full version of the paper. ◀

Simple reductions show that these problems are all NP-hard:

▶ **Proposition 2.** *Membership, vector reachability and* 0*-reachability are all* NP*-hard, even for matrices with only* 1*s on the diagonal, and for $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and for $\vec{y} = \begin{pmatrix} t \\ 1 \end{pmatrix}$ (for vector reachability) resp. $\vec{y}^T = \begin{pmatrix} 1 & -t \end{pmatrix}$ (for* 0*-reachability).*

Proof. The following problem, *multi-subset-sum*, is known to be NP-complete; see the comment under "[MP10] Integer Knapsack" in [13]: given a finite set $\{a_1, \ldots, a_k\} \subseteq \mathbb{N}$ and a value $t \in \mathbb{N}$, decide whether there exist coefficients $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} \alpha_i a_i = t$. Given an instance of multi-subset-sum, construct

$$\mathcal{M} \;:=\; \left\{ \begin{pmatrix} 1 & a_i \\ & 1 \end{pmatrix} \;\middle|\; i \in \{1, \ldots, k\} \right\} \quad \text{and} \quad T \;:=\; \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$

Using the observation that $\begin{pmatrix}1 & a\\ & 1\end{pmatrix}\begin{pmatrix}1 & b\\ & 1\end{pmatrix} = \begin{pmatrix}1 & a+b\\ & 1\end{pmatrix}$ and that, hence, $\langle\mathcal{M}\rangle$ is commutative, it is straightforward to verify that the instance of multi-subset-sum is positive if and only if $T \in \langle\mathcal{M}\rangle$. The proofs for vector reachability and 0-reachability are similar. ◀

On various occasions, we make use of the notion of *polynomial register machine*: Let $\mathbb{Z}[x]$ denote the set of polynomials over x with integer coefficients. A *polynomial register machine (PRM)* is a tuple $\mathsf{R} = (Q, \Delta, \lambda)$ where Q is a finite set of *states*, $\Delta \subseteq Q \times Q$ is the *transition relation*, and $\lambda\colon \Delta \to \mathbb{Z}[x]$ is the *transition labelling function*, labelling each transition with an *update polynomial*. We write $q \xrightarrow{p(x)} q'$ whenever $(q, q') \in \Delta$ and $\lambda((q, q')) = p(x)$. The set $\mathcal{C}(\mathsf{R})$ of *configurations* of R is $\mathcal{C}(\mathsf{R}) := Q \times \mathbb{Z}$. By $(\to_\mathsf{R}) \subseteq \mathcal{C}(\mathsf{R}) \times \mathcal{C}(\mathsf{R})$ we denote the *one-step relation* given by

$$(q, a) \to_\mathsf{R} (q', b) \quad\iff\quad q \xrightarrow{p(x)} q' \text{ and } b = p(a).$$

Let $\to_\mathsf{R}^*$ be the reflexive-transitive closure of $\to_\mathsf{R}$. The next theorem is the main result of [12]:

▶ **Theorem 1** ([12]). *The following problem is* PSPACE*-complete: given a PRM* R *and two configurations* $(q, a), (q', b) \in \mathcal{C}_\mathsf{R}$*, does* $(q, a) \to_\mathsf{R}^* (q', b)$ *hold?*

Restricting register values to positive numbers up to a given bound and update polynomials to simple increments/decrements leads to the notion of a *bounded counter automaton.* A bounded one-counter automaton can be specified as a tuple (Q, b, Δ) where

- Q is a finite set of states,
- $b \in \mathbb{N}$ is a global counter bound,
- Δ is the transition relation containing tuples of the form (q, p, q') where
  - $q, q' \in L$ are predecessor/successor states,
  - $p \in [-b, b]$ specifies how the counter should be modified.

A *configuration* of the automaton consists of a state $q \in Q$ and counter value c. We define the set of configurations to be $S = Q \times [0, b]$. For two configurations $(q, c), (q', c')$ we write $(q, c) \to (q', c')$ whenever $(q, p, q') \in \Delta$ for some $p \in \mathbb{Z}$ such that $c' = c + p \in [0, b]$.

By $\xrightarrow{*}$ we denote the reflexive-transitive closure of the relation $\to$.

▶ **Theorem 2** ([11]). *The reachability problem for bounded one-counter automata is* PSPACE*-complete. This is the following problem: given a bounded one-counter automaton* (Q, b, Δ)*, a state* q_0*, and a configuration* $(q, c) \in S$*, does* $(q_0, 0) \xrightarrow{*} (q, c)$ *hold?*

3 Reachability in Affine Register Machines

In this section, we show that the reachability problem for PRMs is PSPACE-complete even when the register updates are restricted to affine functions. We call such PRMs *affine register machines.* We show that reachability in affine register machines is PSPACE-complete via a reduction from the reachability problem for bounded one-counter automata.

▶ **Theorem 3.** *The following problem is* PSPACE*-complete: Given an affine register machine, and configurations* $(q, x), (r, y)$*, does* $(q, x) \xrightarrow{*} (r, y)$ *hold?*

Proof. Membership in PSPACE follows from Theorem 1. It remains to show that the problem is PSPACE-hard. Fix a bounded one-counter automaton $\mathcal{A} = (Q, b, \Delta)$. We give a polynomial-time construction of affine register machine R such that $(q_0, 0) \xrightarrow{*}_\mathcal{A} (q, c_{\mathrm{tgt}})$ holds for some configurations $(q_0, 0), (q, c_{\text{tgt}})$ of $\mathcal{A}$ if and only if $(r_0, 0) \xrightarrow{*}_{\mathsf{R}} (r, c_{\text{tgt}})$ holds for some distinct states r_0 and r of R.

Let $i \in [0, b]$ and $c \in \mathbb{Z}$, and define:

$$K := 2b + 1 \qquad\qquad K(i, c) := (K + 1)c - i \cdot K.$$

It can be shown that the following implications hold:

$$i \neq c \implies K(i, c) \notin [-b, 2b] \tag{1}$$

$$i = c \implies K(i, c) = i = c \in [0, b] \tag{2}$$

The derivation of these implications can be found in the full version of the paper.

Implications (1) and (2) suggest the following (tentative) construction of the PRM R with affine updates: R stores the counter value of the bounded one-counter automaton $\mathcal{A}$ in its register x, and it stores the state of $\mathcal{A}$ in its state. When R simulates a transition $(q, c) \to (q', c')$ due to $(q, p, q') \in \Delta$, it does the following: It guesses $i \in [0, b]$, and performs the updates $x \leftarrow (K + 1) \cdot x$, followed by the update $x \leftarrow x - i \cdot K$. If $i = c$ and $x \in [0, b]$, then by (2) the register value remains unchanged in the interval $[0, b]$; otherwise the updates result in a value outside the interval $[-b, 2b]$ by (1). Finally, R performs the update $x \leftarrow x + p$ and transitions to state q'. Since $p \in [-b, b]$, our sequence of updates maintains the following invariant: once the register value x lies outside the interval $[0, b]$, it remains so forever. Moreover, if the target state (q, c_{tgt}) is reachable from $(q_0, 0)$ in the counter machine, then we have a corresponding sequence in the affine register machine, and vice versa.

There is one caveat: representing all possible guesses of $i = 0, 1, \ldots, b$ directly in the transition relation of R would not be polynomial. However, these nondeterministic updates can be represented more succinctly with a slight modification: Let $j = \lceil \log b \rceil + 1$. We first transform the counter machine $\mathcal{A}$ into an equivalent machine $\mathcal{A}'$ with counter bound $B = 2^j - 1 \geq b$ by replacing every transition $(q, p, q') \in \Delta$ with three transitions (q, p, q'_1), $(q'_1, (B - b), q'_2)$, and $(q'_2, -(B - b), q')$ (where q'_1 and q'_2 are auxiliary intermediate locations). Observe that by construction of $\mathcal{A}'$, $(q_0, 0) \xrightarrow{*}_{\mathcal{A}} (q, c)$ holds if and only if $(q_0, 0) \xrightarrow{*}_{\mathcal{A}'} (q, c)$. Moreover, the size of $\mathcal{A}'$ is polynomial in the size of $\mathcal{A}$. In order to prove our hardness result, it thus suffices to construct a PRM R of size polynomial in the size of $\mathcal{A}'$, such that $(q_0, 0) \xrightarrow{*}_{\mathcal{A}'} (q, c)$ holds if and only if $(q_0, 0) \xrightarrow{*}_{\mathsf{R}} (q, c)$ holds. To this end, apply the previous construction of the PRM to $\mathcal{A}'$, but instead of guessing $i \in [0, B]$ directly and computing $x \leftarrow x - i \cdot K$, the PRM uses intermediate auxiliary states $r_0, \ldots r_j$, and transitions

$$r_k \xrightarrow{x \leftarrow x - 2^k \cdot K} r_{k+1} \quad \text{(decrement)} \qquad\qquad r_k \xrightarrow{x \leftarrow x} r_{k+1} \quad \text{(no update)}$$

for every $k \in [0, j - 1]$. Thus, instead of guessing i and subsequent decrementation of x by $i \cdot K$, the machine guesses the j binary digits of i in increasing order, and updates the register accordingly after each guess. These nondeterministic choices of binary digits represent all updates for values of i in the range $[0, B] = [0, 2^j - 1]$. The number of states of the resulting PRM is polynomial in the size of the reachability query for $\mathcal{A}'$. This completes the proof. ◀

4 Mortality

In this section we consider the *mortality* problem: given a finite set $\mathcal{M} \subseteq \mathbb{Z}^{2\times 2}$ of integer matrices (not necessarily triangular), is the zero matrix $\mathbf{0}$ in $\langle \mathcal{M} \rangle$? In the upper-triangular case the problem is almost trivial: if there is $M \in \langle \mathcal{M} \rangle$ with only zeros on the diagonal, there must be $M_1, M_2 \in \mathcal{M}$ with $M_1 = \begin{pmatrix} 0 & b \\ & c \end{pmatrix}$ and $M_2 = \begin{pmatrix} a' & b' \\ & 0 \end{pmatrix}$ – but then $M_1 M_2 = \mathbf{0}$. We consider mortality for matrices with determinants $+1, 0$ and prove:

▶ **Theorem 4.** *The mortality problem for two-dimensional integer matrices (not necessarily triangular) with determinants +1 or 0 is* NP*-complete. It is* NP*-hard even if there is one singular matrix and the non-singular matrices are of the form* $\begin{pmatrix}1 & a\\ & 1\end{pmatrix}$.

Both for the lower and the upper bound we need the following lemma from [6]:

▶ **Lemma 5** ([6, Lemma 2])**.** *Let* $\mathcal{M} \subseteq \mathbb{R}^{2\times 2}$ *be a finite set of matrices. We have* $\mathbf{0} \in \langle \mathcal{M} \rangle$ *if and only if there are* $M_1, \ldots, M_n \in \mathcal{M}$ *with* $M_1 \cdots M_n = \mathbf{0}$ *and* $rank(M_1) = rank(M_n) < 2$ *and* $rank(M_i) = 2$ *for all* $i \in \{2, \ldots, n-1\}$.

First we prove NP-hardness:

Proof of the NP-hardness part of Theorem 4. Concerning NP-hardness, the reduction from Proposition 2 for 0-reachability constructs, given an instance of multi-subset-sum, a set $\mathcal{M}$ of matrices of the form $\begin{pmatrix}1 & a\\ & 1\end{pmatrix}$ (hence of rank 2) and a number $t \in \mathbb{N}$ such that the instance of multi-subset-sum is positive if and only if there is $M \in \mathcal{M}$ with $\begin{pmatrix}1 & -t\end{pmatrix} M \begin{pmatrix}0\\1\end{pmatrix} = 0$. Define the rank-1 matrix $T := \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & -t\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & -t\end{pmatrix}$ and set $\mathcal{M}' := \mathcal{M} \cup \{T\}$. If there is $M \in \mathcal{M}$ with $\begin{pmatrix}1 & -t\end{pmatrix} M \begin{pmatrix}0\\1\end{pmatrix} = 0$, then $\mathbf{0} = TMT \in \langle \mathcal{M}' \rangle$. Conversely, if there is $\mathbf{0} \in \langle \mathcal{M}' \rangle$, then, by Lemma 5, there is $M \in \mathcal{M}$ with $TMT = \mathbf{0}$, hence $\begin{pmatrix}1 & -t\end{pmatrix} M \begin{pmatrix}0\\1\end{pmatrix} = 0$. ◀

We remark that this NP-hardness proof subsumes the main result of [1], which is slightly weaker in that it allows also for matrices with determinant −1. For the upper bound we use results from [24, 2]. As usual, define $\mathsf{SL}(2,\mathbb{Z}) := \{M \in \mathbb{Z}^{2\times 2} \mid \det(M) = 1\}$.

▶ **Lemma 6** ([24, Lemma 4])**.** *Let* $\vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{Z}^2$ *and* $\vec{y} = \begin{pmatrix}y_1\\y_2\end{pmatrix} \in \mathbb{Z}^2$ *and* $M \in \mathsf{SL}(2,\mathbb{Z})$. *If* $M\vec{x} = \vec{y}$ *then* $\gcd(x_1, x_2) = \gcd(y_1, y_2)$.

▶ **Theorem 7** ([24, Theorem 8, Corollary 9])**.** *Let* $\vec{x}, \vec{y} \in \mathbb{Z}^2$ *with* $\vec{x} \neq \vec{0}$. *Then one can compute in polynomial time matrices* $B, C \in \mathsf{SL}(2,\mathbb{Z})$ *such that for every* $M \in \mathsf{SL}(2,\mathbb{Z})$ *the following equivalence holds:* $M\vec{x} = \vec{y} \iff$ *there is* $k \in \mathbb{Z}$ *with* $M = B\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^k C$.

In the following theorem, a *regular expression* describes a set of matrices, so that the atomic expressions describe singleton sets, the operator ∪ is set union, and the operator · is elementwise multiplication:

▶ **Theorem 8** ([2, Corollary 5.1])**.** *Given a regular expression over matrices in* $\mathsf{SL}(2,\mathbb{Z})$*, one can decide in* NP *whether the set described by the regular expression intersects with* $\{I, -I\}$*, where* I *denotes the identity matrix.*

Now we can complete the proof of Theorem 4:

Proof of the upper bound in Theorem 4. We give an NP procedure. We guess the matrices M_1, M_n from Lemma 5. Define $\mathcal{M}' := \mathcal{M} \cap \mathsf{SL}(2,\mathbb{Z})$. We have to verify that there is a matrix $M \in \langle \mathcal{M}' \rangle$ such that $M_1 M M_n = \mathbf{0}$. Let $\vec{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{Z}^2$ be a non-zero rational multiple of a non-zero column of M_n (if M_n does not have a non-zero column, the problem is trivial) such that $\gcd(x_1, x_2) = 1$. This defines $\vec{x}$ uniquely up to a sign. Similarly, let $\begin{pmatrix} y_1 & y_2 \end{pmatrix} \in \mathbb{Z}^2$ be a non-zero rational multiple of a non-zero row of M_1 such that $\gcd(y_1, y_2) = 1$. Now it suffices to check whether there is $M \in \langle \mathcal{M}' \rangle$ such that $\begin{pmatrix} y_1 & y_2 \end{pmatrix} M\vec{x} = 0$. By Lemma 6, this holds if and only if $M\vec{x} \in \{\vec{y}, -\vec{y}\}$ where $\vec{y} := \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$. For $\vec{y}$ and $-\vec{y}$, compute the matrices B_1, C_1 and B_2, C_2 from Theorem 7, respectively. Let $\{A_1, \ldots, A_m\} = \mathcal{M}'$, and note that $A_1^{-1}, \ldots, A_m^{-1} \in \mathsf{SL}(2, \mathbb{Z})$.

We claim that there is $M \in \langle \mathcal{M}' \rangle$ with $M\vec{x} \in \{\vec{y}, -\vec{y}\}$ if and only if there is $i \in \{1, 2\}$ with

$$B_i \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right)^* C_i \left(A_1^{-1} \cup \cdots \cup A_m^{-1} \right)^* \cap \{I, -I\} \quad \neq \quad \emptyset. \tag{3}$$

By Theorem 8, this claim completes the proof.

To prove the claim, suppose there is $M \in \langle \mathcal{M}' \rangle$ with $M\vec{x} = +\vec{y}$ (the negative case is similar). By Theorem 7, there is $k \in \mathbb{Z}$ with $I = B_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k C_1 M^{-1}$, implying (3). Conversely, suppose (3) holds for $i = 1$ (the case $i = 2$ is similar). Then there are $k \in \mathbb{Z}$ and $M \in \langle \mathcal{M}' \rangle$ such that $B_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k C_1 \in \{M, -M\}$. By Theorem 7 it follows that $M\vec{x} = \vec{y}$ or $-M\vec{x} = \vec{y}$. ◀

5 Two-Dimensional Upper-Triangular Integer Matrices

Motivated by the connections with affine reachability (Proposition 1) we study in this section the complexity of reachability problems in finitely generated monoids $\langle \mathcal{M} \rangle$ over two-dimensional upper-triangular integer matrices. Specifically, we consider membership, vector reachability, and scalar reachability as defined in Section 2.

5.1 Determinant ± 1

In this section we study the case where the monoid $\langle \mathcal{M} \rangle$ is restricted to matrices with determinants ± 1, i.e., with ± 1 on the diagonal. In this case, the matrices $M \in \langle \mathcal{M} \rangle$ are characterized by the sign pattern on the diagonal and the top-right entry. Our problems become NP-complete under this restriction, but are in P if the determinants are -1. First we prove the following lemma:

▶ **Lemma 9.** *Let $\mathcal{M} \subseteq \mathsf{UT}$ be with $\det(M) \in \{1, -1\}$ for all $M \in \mathcal{M}$. There exists an existential Presburger formula $\varphi(s, a, t)$ that can be constructed in time polynomial in the description of $\mathcal{M}$ such that $\varphi(s, a, t)$ holds if and only if $\begin{pmatrix} s & a \\ & t \end{pmatrix} \in \langle \mathcal{M} \rangle$.*

Proof sketch. Note that $\mathcal{M} \subseteq \mathsf{UT}[|A_{11}| = |A_{22}| = 1]$. We reduce the problem whether $\begin{pmatrix} s & a \\ & t \end{pmatrix} \in \langle \mathcal{M} \rangle$ holds for some $s, t \in \{1, -1\}$ and $a \in \mathbb{Z}$ to a reachability problem on one-dimensional $\mathbb{Z}$-*VASS* (integer vector addition systems with states) [15]. The reachability relation of one-dimensional $\mathbb{Z}$-VASS is known to be effectively definable by an existential Presburger formula in polynomial time; see, e.g., [15]. This entails the claim to be shown. See the full version of the paper for the details. ◀

▶ **Theorem 10.** *Let $\mathcal{M} \subseteq \mathsf{UT}$ be with $\det(M) \in \{1, -1\}$ for all $M \in \mathcal{M}$.*

1. *Membership, vector reachability and scalar reachability are NP-complete.*
2. *They are NP-hard even for $\mathcal{M} \subseteq \mathsf{UT}[A_{11} = A_{22} = 1]$ and for $\mathcal{M} \subseteq \mathsf{UT}[A_{11} = A_{22} = -1]$.*
3. *They are in P if $\det(M) = -1$ for all $M \in \mathcal{M}$.*

Proof (sketch). For item 1 the lower bound follows from Proposition 2. The upper bound for membership follows from Lemma 9 and the folklore result that existential Presburger arithmetic is in NP [5, 28]. Vector and scalar reachability are easily reduced to membership, as there are only four choices in total for the diagonal entries s, t, and this choice together with the input determines the top-right entry uniquely. This completes the proof of item 1.

Towards item 2, NP-hardness of the case $\mathsf{UT}[A_{11} = A_{22} = 1]$ follows from the proof of Proposition 2. For the case $\mathsf{UT}[A_{11} = A_{22} = -1]$ we adapt this reduction by constructing

$$\mathcal{M} := \left\{ \begin{pmatrix} -1 & -a_i \\ & -1 \end{pmatrix} \,\middle|\, i \in \{1, \ldots, k\} \right\} \cup \left\{ \begin{pmatrix} -1 & 0 \\ & -1 \end{pmatrix} \right\} \quad \text{and} \quad T := \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$

Note that $\begin{pmatrix} -1 & -a \\ & -1 \end{pmatrix} \begin{pmatrix} -1 & -b \\ & -1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ & 1 \end{pmatrix}$. The additional (negative identity) matrix ensures that an even number of matrices from $\mathcal{M}$ can be used to form the product T. NP-hardness for vector reachability and 0-reachability are similar. This completes the proof of item 2.

Towards item 3, we will give an explicit description of $\langle \mathcal{M} \rangle$, such that membership can be checked in polynomial time. In slightly greater detail, we focus on matrix products of even length $M_1 \cdots M_{2n} \in \langle \mathcal{M} \rangle$. These are exactly the matrices in $\langle \mathcal{M} \rangle$ with determinant 1. The extension to odd-length products (which have determinant -1) will be straightforward, as such products simply arise from even-length products multiplied with a single element of $\mathcal{M}$. The even-length products also form a monoid, finitely generated by $\mathcal{M}' := \{M_1 M_2 \mid M_1 \in \mathcal{M},\ M_2 \in \mathcal{M}\}$, and all matrices in $\mathcal{M}'$ have $(+1, +1)$ or $(-1, -1)$ on the diagonal. Clearly, $\mathcal{M}'$ can be computed in polynomial time. We show in the full version of the paper that $\langle \mathcal{M}' \rangle$ can be characterized by a system of affine Diophantine equations. It is known that affine Diophantine equations can be solved in polynomial time [27, Chapter 5]. The vector reachability and scalar reachability problems (with the restriction on determinants in place) easily reduce to the membership problem, hence are also in P. ◀

5.2 Vector Reachability

We show:

▶ **Theorem 11.** *The vector reachability problem for $\mathsf{UT}[A_{22} \neq 0]$ is in PSPACE.*

Proof. We construct a nondeterministic Turing machine $\mathcal{T}$ that decides the reachability problem for $\mathsf{UT}[A_{22} \neq 0]$ in polynomial space. Let $\mathcal{M} \subseteq \mathsf{UT}[A_{22} \neq 0]$ and $\vec{x}, \vec{y} \in \mathbb{Z}^2$ be an instance of the reachability problem, that is, $\mathcal{T}$ has to check whether $M \cdot \vec{x} = \vec{y}$ holds for some $M \in \langle \mathcal{M} \rangle$.

Assume that $M^{(1)} \cdot \ldots \cdot M^{(k)} \cdot \vec{x} = \vec{y}$ holds for some $M^{(1)}, \ldots, M^{(k)} \in \mathcal{M}$. Observe that for all $A \in \mathcal{M}$ and all $z_1, z_2, z_1', z_2' \in \mathbb{Z}$ such that $A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix}$, we have $z_2' = A_{22} z_2$. From this observation we conclude:

1. If $x_2 \neq 0$, then $y_2 \neq 0$, too, and the number of indices $1 \leq i \leq k$ s.t. $|M_{22}^{(i)}| > 1$ is bounded by $\mathcal{O}(\log(|y_2|))$.
2. If $x_2 = 0$, then $y_1 = M_{11}^{(1)} \cdot \ldots \cdot M_{11}^{(k)} \cdot x_1$.
3. If $x_2 = 0$ or $y_2 = 0$, then $x_2 = y_2 = 0$.

Let us first consider the case where $x_2 = 0$ or $y_2 = 0$ holds. In this case, $\mathcal{T}$ rejects the input if $x_2 \neq 0$ or $y_2 \neq 0$. Otherwise, $\mathcal{T}$ needs to check whether y_1 can be written as a product $M_{11}^{(1)} \cdot \ldots \cdot M_{11}^{(k)} \cdot x_1$ for some indices $1, \ldots, k$, which can be done in polynomial space (even in NP), since k can be bounded by $\mathcal{O}(\log(|y_1|))$.

Now consider the case where $|x_2| > 0$ and $|y_2| > 0$ holds. By the above observation, if the reachability problem has a solution, it can be given by

$$\vec{y} = A^{(l+1)} \cdot B^{(l)} \cdot A^{(l)} \cdot \ldots \cdot B^{(2)} \cdot A^{(2)} \cdot B^{(1)} \cdot A^{(1)} \cdot \vec{x}\ , \tag{4}$$

- where the length of $l \in \mathbb{N}$ is polynomially bounded in the size of the input,
- $B^{(i)} \in \mathsf{UT}[|A_{22}| > 1] \cap \mathcal{M}$ for every i,
- $A^{(i)}$ can be written as product of matrices from $\mathsf{UT}[|A_{22}| = 1] \cap \mathcal{M}$.

Notice that the matrices from $\mathsf{UT}[|A_{22}| = 1]$ behave like affine update polynomials in a PRM, with the register value stored in the first component of the vector. This suggests the following approach: $\mathcal{T}$ guesses the sequence of matrices $B = B^{(1)}, \ldots, B^{(l)}$ and constructs a PRM R_B, whose size is polynomially bounded in the size of the input, such that $(q, x_1) \to_{\mathsf{R}_B}^{*} (q', y_1)$ holds for some fixed states q, q' if the reachability problem has a solution of the form given in (4). The register machine only needs to store in its states how many of the B-matrices have already been applied, plus the current sign of the second vector component reached thus far. The size is polynomial in the input. The claim then follows by applying Theorem 1. Details can be found in the full version of the paper. ◀

Without the restriction on $\mathsf{UT}[A_{22} \neq 0]$ the vector reachability problem becomes hard for affine reachability over $\mathbb{Q}$:

▶ **Theorem 12.** *There is a polynomial-time Turing reduction from affine reachability over $\mathbb{Q}$ to vector reachability.*

Proof. Let an instance of affine reachability over $\mathbb{Q}$ be given. We first assume that all input functions are non-constant. Then we use the reduction from Proposition 1.3 to obtain an instance of the 0-reachability problem: $\mathcal{M} \subseteq \mathsf{UT}[A_{11} \neq 0,\ A_{22} \neq 0]$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $x_2 \neq 0$ and $y_1 \neq 0$. (The top-left entries are non-zero as the functions are non-constant.) Define $T := \begin{pmatrix} y_1 & y_2 \\ & 0 \end{pmatrix}$ and $\mathcal{M}' := \mathcal{M} \cup \{T\}$. We show that the instance of the 0-reachability problem is positive if and only if the vector reachability for $\mathcal{M}'$ and $\vec{x}$ and $\vec{0}$ is positive. Suppose the instance of the 0-reachability problem is positive. Then there is $M \in \langle \mathcal{M} \rangle$ such that $\begin{pmatrix} y_1 & y_2 \end{pmatrix} M\vec{x} = 0$, thus $TM\vec{x} = \vec{0}$, so $\vec{0}$ is reachable from $\vec{x}$. Conversely, suppose the instance of the 0-reachability problem is negative. Let $M \in \langle \mathcal{M} \rangle$. Then $M\vec{x} \neq \vec{0}$, as the bottom component of $M\vec{x}$ is non-zero. Since the instance of the 0-reachability problem is negative, we have $TM\vec{x} = \begin{pmatrix} t \\ 0 \end{pmatrix}$ for some $t \neq 0$. Since the top-left component of all matrices in $\mathcal{M}' \supseteq \{T\}$ is non-zero, it follows that $M'TM\vec{x} \neq \vec{0}$ holds for all $M' \in \langle \mathcal{M}' \rangle$. Thus, $M''\vec{x} \neq \vec{0}$ holds for all $M'' \in \langle \mathcal{M}' \rangle$, and so the instance of the vector reachability problem is negative.

Now we allow input functions of affine reachability to be constant. Suppose the constant functions are $f_i : x \mapsto 0x + c_i$ for $i \in \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. It is easy to see that then the affine reachability problem can be solved by removing all f_i from the set of functions and checking affine reachability starting from c_i, where $i \in \{1, \ldots, n\}$, one by one. These instances can be reduced to vector reachability, as described before. ◀

5.3 Membership

In this section we study the membership problem. As we will see, the difficulty depends on how many 0s are allowed on the diagonal. Any product of upper-triangular matrices is non-zero on the top-left (bottom-right, respectively) if and only if all factors are non-zero on the top-left (bottom-right, respectively). So when we speak of the membership problem for, say, $\mathsf{UT}[A_{11} \neq 0]$, the restriction refers both to $\mathcal{M}$ and the target matrix T.

The case with no 0s on the diagonal is NP-complete:

▶ **Theorem 13.** *The membership problem for* $\mathsf{UT}[A_{11} \neq 0 \wedge A_{22} \neq 0]$ *is* NP*-complete.*

Proof. The lower bound was shown in Proposition 2. For the upper bound, we construct an NP Turing machine. Fix $\mathcal{M}$ and T. Assume for the moment that T can be written as a product $T = M^{(k)} \cdot \ldots \cdot M^{(1)}$ of matrices from $\mathcal{M}$. Let l be the number of indices $i > 1$ where $M^{(i)} \in \mathsf{UT}[|A_{11}| > 1 \vee |A_{22}| > 1]$ holds. Since $T_{ii} = \prod_{j=1}^{k} M_{ii}^{(j)}$ holds for both $i \in \{1, 2\}$, the number l can be bounded by $\mathcal{O}(\log(|T_{11}|) + \log(|T_{22}|))$, and T can be written as

$$T = A^{(l+1)} B^{(l)} A^{(l)} \cdot \ldots \cdot B^{(1)} \cdot A^{(1)} M^{(1)} \tag{5}$$

s.t. $B^{(j)} \in \mathsf{UT}[|A_{11}| > 1 \vee |A_{22}| > 1] \cap \mathcal{M}$ and $A^{(j)} \in \langle \mathsf{UT}[|A_{11}| = |A_{22}| = 1] \cap \mathcal{M} \rangle$ for all j.

The Turing machine guesses matrices $B^{(1)}, \ldots, B^{(l)}$ and the matrix $M^{(1)}$ and constructs in polynomial time an existential Presburger formula $\varphi(t_1, t_2, t_3)$ that satisfies $t_1 = T_{11}$, $t_2 = T_{12}$, $t_3 = T_{22}$ if and only if T can be written as a product of the form given in (5) for the guessed $B^{(i)}$ and $M^{(1)}$. By Lemma 9, such a formula $\varphi(t_1, t_2, t_3)$ exists and can be efficiently constructed. The claim then follows from the fact that $\varphi(t_1, t_2, t_3)$ is existential Presburger of size polynomial in the input, and that the existential Presburger fragment is in NP [5, 28]. ◀

The proof of Proposition 1.1 with the isomorphism between affine functions over $\mathbb{Z}$ and upper-triangular matrices with 1 on the bottom-right shows that non-constant functions correspond to matrices that do not have 0s on the diagonal. Hence we have:

▶ **Corollary 14.** *Affine membership over* $\mathbb{Z}$ *with non-constant functions is* NP*-complete.*

The case with at most one 0 on the diagonal can be reduced to vector reachability:

▶ **Theorem 15.** *The membership problems for* $\mathsf{UT}[A_{11} \neq 0]$ *and for* $\mathsf{UT}[A_{22} \neq 0]$ *are in* PSPACE*.*

Proof. We give a proof sketch for $\mathsf{UT}[A_{22} \neq 0]$; the detailed proof can be found in the full version of the paper. If $T_{11} \neq 0$, then a PSPACE decision procedure follows from Theorem 13. If $T_{11} = 0$, then the problem reduces to a reachability problem with the additional restriction that some element of $\mathsf{UT}[A_{11} = 0]$ must be included in the matrix product. This problem in turn is decidable in PSPACE via a straightforward modification of the PRM R_B in the proof of Theorem 11. ◀

The general membership problem, without restrictions on the position of 0s, is related to (variants of) scalar reachability. Theorems 16 and 17 provide reductions in both ways.

▶ **Theorem 16.** *Let s be an oracle for the scalar reachability problem. The membership problem is in* PSPACE^s*.*

Proof. Fix some finite $\mathcal{M} \subseteq \mathsf{UT}$ and $T \in \mathsf{UT}$. We give a PSPACE^s procedure that decides whether $T \in \langle \mathcal{M} \rangle$ holds. We make the following case distinction:

1. $T = \mathbf{0}$
2. $T \in \mathsf{UT}[A_{11} \neq 0 \vee A_{22} \neq 0]$
3. $T \in \mathsf{UT}[A_{11} = A_{22} = 0 \wedge A_{12} \neq 0]$

In the first case, the membership problem is easy: if $T = \mathbf{0} \in \langle \mathcal{M} \rangle$, then there must exist matrices $M_1 \in \mathsf{UT}[A_{11} = 0] \cap \mathcal{M}$ and $M_2 \in \mathsf{UT}[A_{22} = 0] \cap \mathcal{M}$, but then $T = \mathbf{0} = M_1 \cdot M_2$. The existence of such M_1, M_2 is trivial to check. In the second case, the problem reduces to $T \in \langle \mathsf{UT}[A_{11} \neq 0] \cap \mathcal{M} \rangle$ or $T \in \langle \mathsf{UT}[A_{22} \neq 0] \cap \mathcal{M} \rangle$, which is decidable in PSPACE by Theorem 15. In the full version of the paper we show that the last case reduces to an instance of scalar reachability by an NP procedure, and thus can be solved in $\mathsf{NP}^s \subseteq \mathsf{PSPACE}^s$. ◀

▶ **Theorem 17.** *The following sign-invariant version of the scalar-reachability problem is polynomial-time Turing-reducible to the membership problem: given $\mathcal{M} \subseteq \mathsf{UT}$ and column vectors $\vec{x}, \vec{y} \in \mathbb{Z}^2$, does $\vec{y}^T M \vec{x} \in \{-1, 1\}$ hold for some $M \in \langle \mathcal{M} \rangle$?*

Proof. Fix $\mathcal{M}, \vec{x}, \vec{y}$. Let I be the identity, $\mathcal{A} := \mathcal{M} \cap \mathsf{UT}[A_{22} = 0]$, $\mathcal{B} := \mathcal{M} \cap \mathsf{UT}[A_{11} = 0]$, $\mathcal{C} := (\mathcal{M} \setminus (\mathcal{A} \cup \mathcal{B}))$, $Y := \begin{pmatrix} y_1 & y_2 \\ 0 & 0 \end{pmatrix}$, and $X := \begin{pmatrix} 0 & x_1 \\ 0 & x_2 \end{pmatrix}$. Set $\mathcal{A}' := \mathcal{A}$, if $|y_1| = 1$, otherwise set $\mathcal{A}' := \emptyset$; further set $\mathcal{B}' := \mathcal{B}$, if $|x_2| = 1$, otherwise set $\mathcal{B}' := \emptyset$.

We obtain the following equivalences:

$$\exists M \in \langle \mathcal{M} \rangle \colon \vec{y}^T M \vec{x} \in \{\pm 1\} \quad \Leftrightarrow \tag{6}$$

$$\exists A \in \mathcal{A} \cup \{I\}, B \in \mathcal{B} \cup \{I\}, C \in \langle \mathcal{C} \rangle \colon \vec{y}^T \cdot (A \cdot C \cdot B) \cdot \vec{x} \in \{\pm 1\} \quad \Leftrightarrow \tag{7}$$

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} \in \bigcup_{A \in \mathcal{A}' \cup \{Y\}, B \in \mathcal{B}' \cup \{X\}} \langle \mathcal{C} \cup \{A, B\} \rangle. \tag{8}$$

We provide detailed derivations of these equivalences in the full version of the paper. Deciding (8) requires polynomially many queries to a membership oracle where input sizes are polynomial in the description of $\mathcal{M}, \vec{x}, \vec{y}$. This entails the theorem. ◀

6 Conclusion

We have proved PSPACE-completeness of reachability in affine register machines, and NP-completeness of the mortality problem over two-dimensional integer matrices with determinants $+1$ or 0.

Motivated by their connections to affine reachability, we have studied membership, vector reachability, and scalar reachability for two-dimensional upper-triangular integer matrices. We have established several complexity results and reductions. Concerning upper complexity bounds, we have employed a variety of techniques, including existential Presburger arithmetic, $\mathbb{Z}$-VASS, PRMs, and solving linear Diophantine equations over the integers. We have also established lower bounds, including hardness of vector reachability for affine reachability over $\mathbb{Q}$, and a connection between membership and scalar reachability.

As open problem, we highlight the precise complexity (between NP and PSPACE) of (stateless) affine reachability over $\mathbb{Z}$.

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